# Classification of topological insulators using Clifford algebras

理化学研究所

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established in 1917


Tomonaga worked in RIKEN in 1930s.

Dirac and Heisenberg visited RIKEN and Univ. of Tokyo in 1930.

# MOU between Tsinghua Univ. and RIKEN

Prof. Q.-k. Xue

A picture from  the signing ceremony on Nov. 13, 2013

RIKEN Center for Emergent Matter Science
Director: Yoshinori Tokura

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# Plan of this talk

- Introduction
  - Some examples of topological insulators/superconductors
- Table of topological insulators and superconductors
  - 10 Altland-Zirnbauer symmetry classes
  - Time-reversal, particle-hole, and chiral symmetries
- Derivation of the periodic table
  - Dirac Hamiltonian
  - Clifford algebras

# Collaborators:

Shinsei Ryu (U Illinois at Urbana-Champaign)

Anderas Schnyder (Max Planck Inst. Stuttgart)

Andreas Ludwig (UC Santa Barbara)

Schnyder, Ryu, AF, and Ludwig, Phys. Rev. B **78**, 195125 (2008)

AIP Conf. Proc. **1134**, 10 (2009) = arXiv:0905.2029

Ryu, Schnyder, AF, and Ludwig, New J. Phys. **12**, 065010 (2010)

Takahiro Morimoto (RIKEN)

Morimoto & AF, Phys. Rev. B 88, 125129 (2013);
arXiv:1310.5862;
a paper in preparation

# Topological insulators

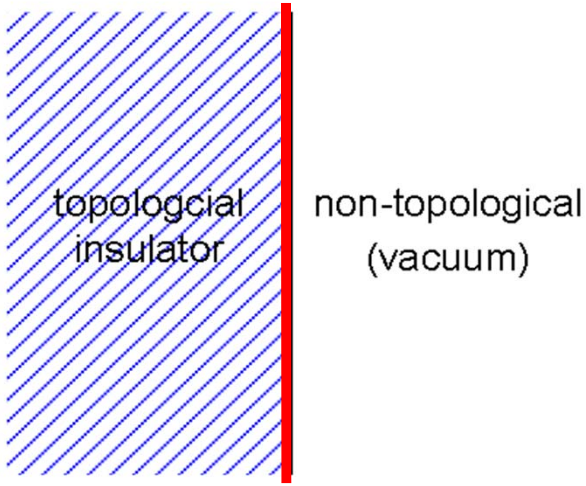
in the broad sense

- band insulators

free fermions (ignore e-e int.)

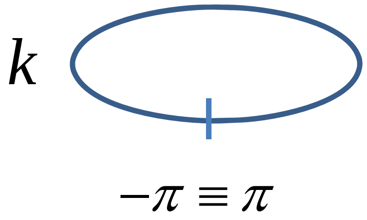
- characterized by a topological number (Z or $Z_2$)
- gapless excitations at boundaries

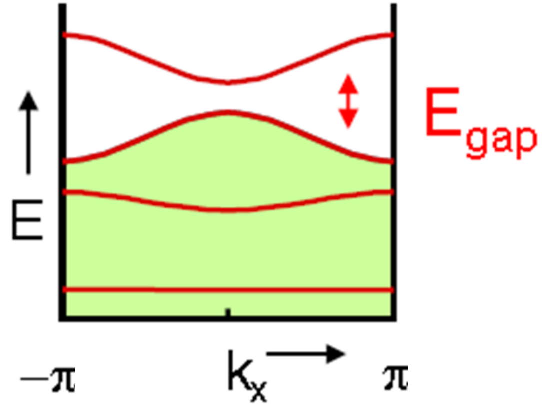
stable



# Energy band structure:

map $\vec{k} \mapsto E_n(\vec{k}), \quad \text{or} \quad \left|u_n(\vec{k})\right\rangle$




topological numbers (e.g., winding number)

Band structures are topologically equivalent,
if they can be continuously deformed into one another
without closing the energy gap.

Topological numbers are not changed by continuous deformation.

(discrete number)

# Topological (band) insulators

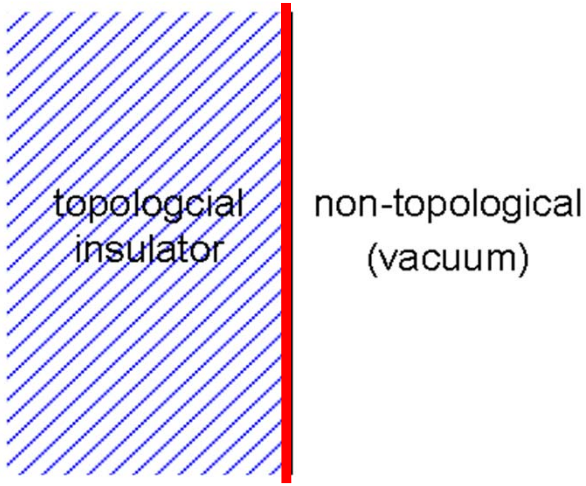
in the broader sense

- band insulators

free fermions (ignore e-e int.)

- characterized by a topological number (Z or $Z_2$)
- gapless excitations at boundaries

stable

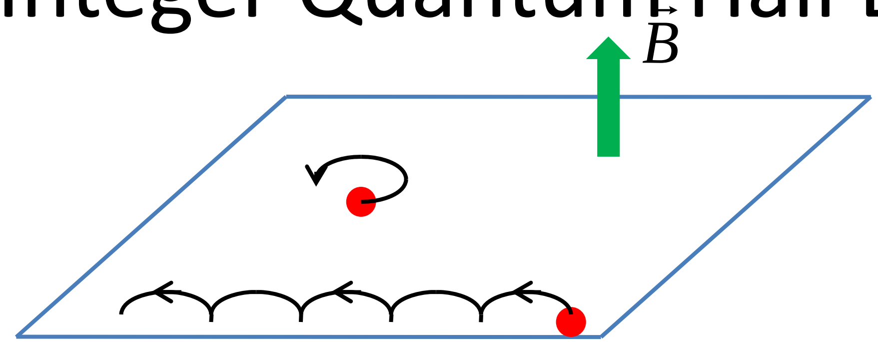


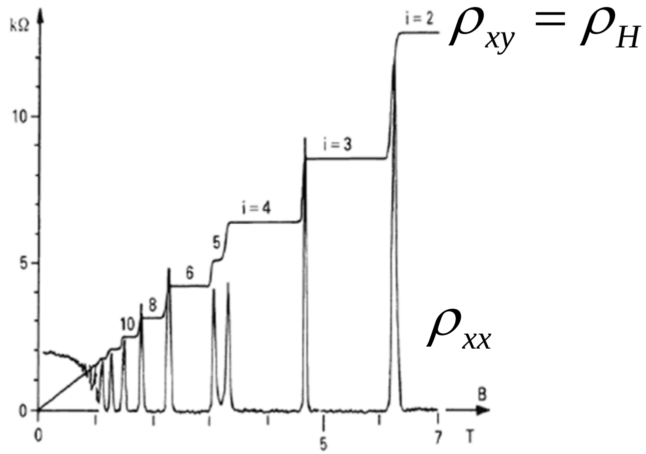

Examples: integer quantum Hall effect,

time reversal symmetry ⟹ quantum spin Hall insulator, 3D $Z_2$ topological insulator, ….

2D 3D

# Integer Quantum Hall Effect




# TKNN number (Thouless-Kohmoto-Nightingale-den Nijs)

$$\sigma_{xy} = -\frac{e^2}{h} C$$

TKNN (1982); Kohmoto (1985)

1st Chern number

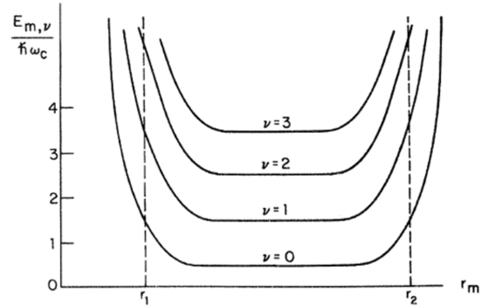
integer valued

$$C = \frac{1}{2\pi i} \int_{\text{filled band}} d^2k \, \vec{\nabla}_k \times \vec{A}(k_x, k_y)$$ = number of edge modes crossing $E_F$

bulk-edge correspondence

$$\vec{A}(k_x, k_y) = \langle \vec{k} | \vec{\nabla}_k | \vec{k} \rangle$$ Berry connection

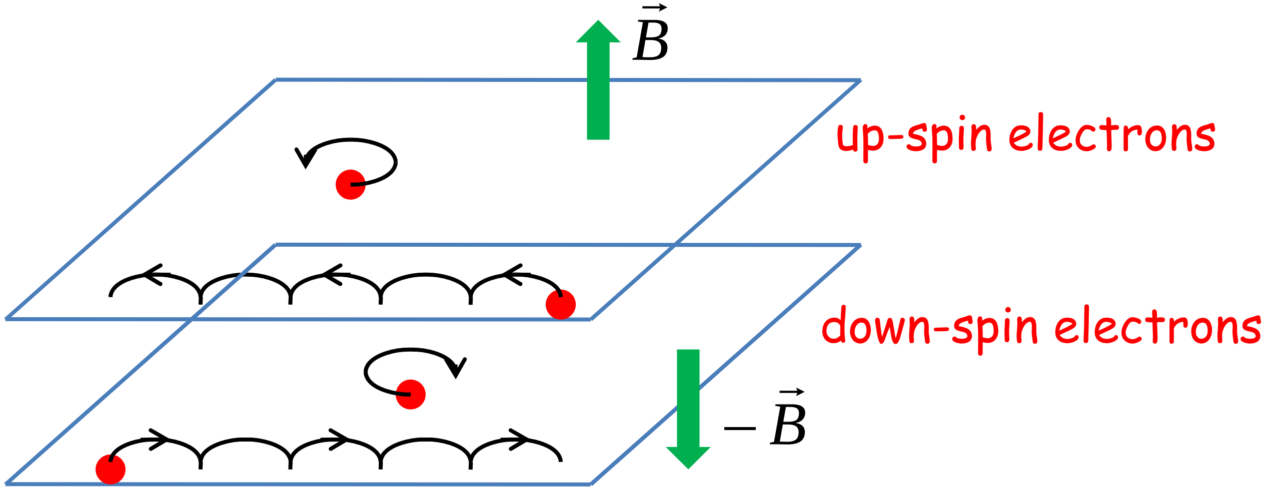
$$\vec{\nabla}_k = (\partial_{k_x}, \partial_{k_y})$$

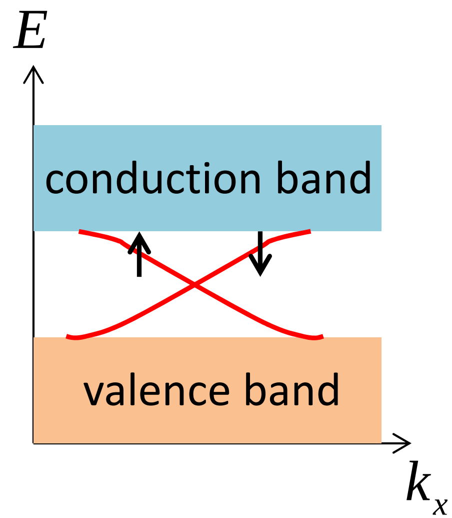


# 2D Quantum spin Hall effect (2D $Z_2$ TPI)

Kane & Mele (2005, 2006); Bernevig & Zhang (2006)

- time-reversal invariant band insulator
- spin-orbit interaction
- gapless helical edge mode (Kramers' pair)



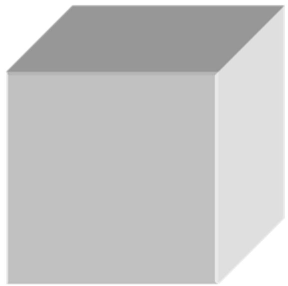




$S^z$ is not conserved in general.

Topological index: Z ⇒ $Z_2$

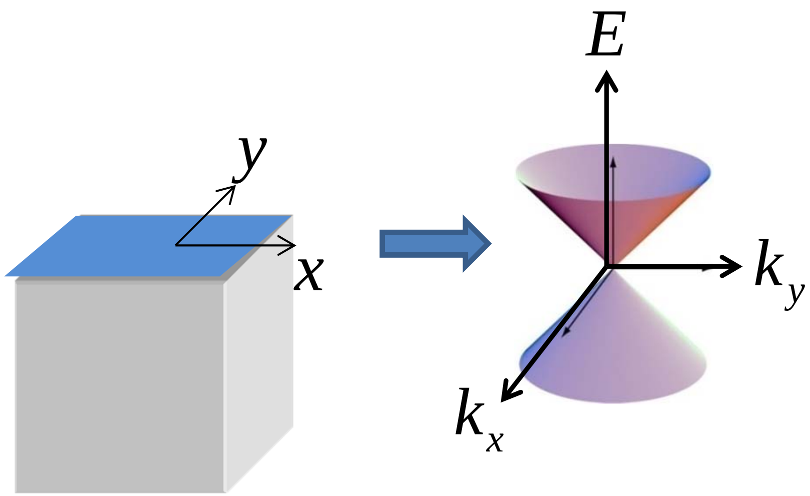
# 3 dimensional $Z_2$ Topological insulator

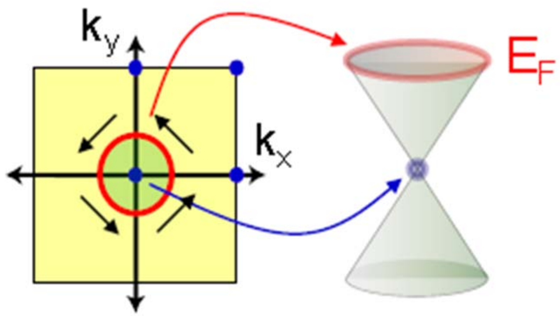
- Band insulator



$Z_2$ topologically nontrivial

- Metallic surface: massless Dirac fermions



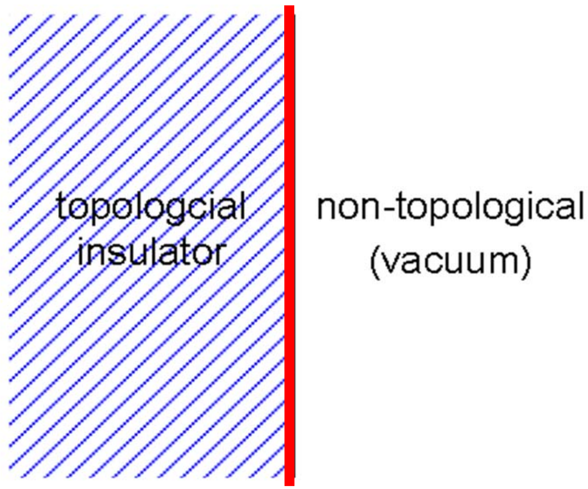

an odd number of Dirac cones/surface

Theoretical Predictions made by:
Fu, Kane, & Mele (2007)
Moore & Balents (2007)
Roy (2007)

# Topological superconductors

- BCS superconductors with a fully gapped Fermi surface
- characterized by a topological number
- gapless excitations at boundaries (Dirac or Majorana)

stable

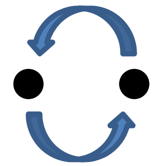



Examples: p+ip superconductor, $^3$He, …

particle-hole symmetry (BdG Hamiltonian)

# 2D p+ip superconductor  $^3$He-A thin film, $Sr_2RuO_4$

- ($p_x$+i$p_y$)-wave Cooper pairing



- Hamiltonian  Nambu-spinor $\begin{pmatrix} c_{\vec{p}} \\ c^\dagger_{-\vec{p}} \end{pmatrix}$ (spinless fermions)

$$H_{\vec{p}} = \begin{pmatrix} \frac{p^2}{2m} - \mu & \frac{\Delta}{p_F}\left(p_x + ip_y\right) \\ \frac{\Delta}{p_F}\left(p_x - ip_y\right) & \mu - \frac{p^2}{2m} \end{pmatrix} = \vec{d}\left(\vec{p}\right)\cdot\vec{\sigma} \qquad \hat{d} = \vec{d}\big/\left|\vec{d}\right|$$

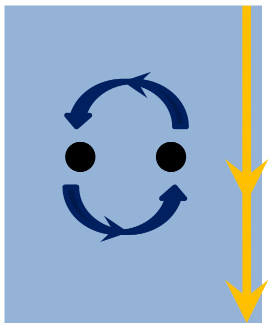
$$S^2$$

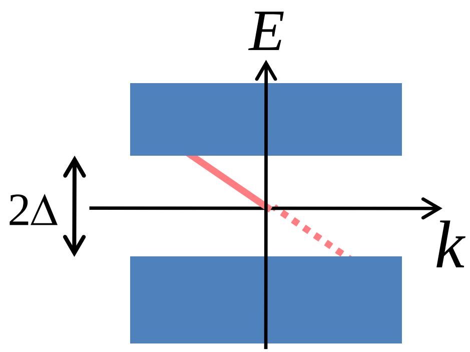
$$\left(p_x, p_y\right) \mapsto S^2$$

wrapping # = 1

- Majorana edge state

$p_x$+i$p_y$



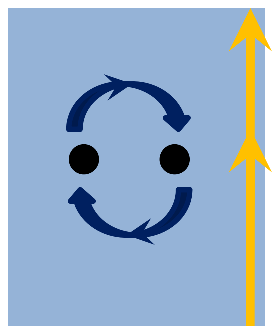



$$\gamma_k = \gamma^\dagger_{-k}$$

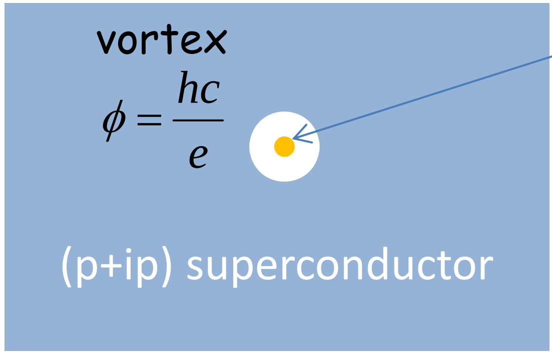
$$\psi\left(x\right) = \int_{k>0} \left(e^{ikx}\gamma_k + e^{-ikx}\gamma^\dagger_k\right) dk$$

$$= \psi^\dagger\left(x\right)$$

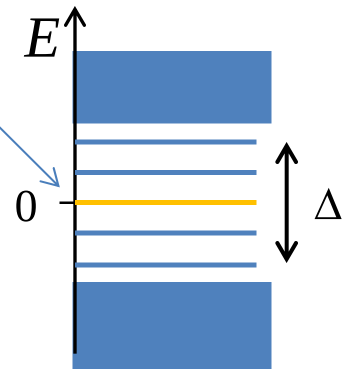
$p_x$-i$p_y$



# Majorana zeromode in a quantum vortex




Zero-energy Majorana bound state




energy spectrum
near a vortex

zero mode $\varepsilon_0 = 0$ $\gamma_0 = \gamma_0^+$

Majorana fermion

If there are 2N vortices, then the ground-state degeneracy = $2^N$.

## 1D p-wave superconductor (Kitaev)

P-wave SC

Majorana fermion

Q: How many classes of topological insulators/superconductors exist in nature?

Topological insulators/superconductors should be stable against arbitrary perturbation (deformation of Hamiltonian) that respects symmetry constraints.

classification based on generic symmetries:

time reversal

charge conjugation (particle hole)  SC

random matrix theory

A: There are 5 classes of TPIs or TPSCs in each spatial dimension.  $3Z$ & $2Z_2$

# Table of topological insulators/superconductors for d=1,2,3

| | 10 Symmetry Classes | TRS | PHS | CS | d=1 | d=2 | d=3 | Annotation |
|---|---|---|---|---|---|---|---|---|
| Standard (Wigner-Dyson) | A (unitary) | 0 | 0 | 0 | -- | Z | -- | IQHE |
| | AI (orthogonal) | +1 | 0 | 0 | -- | -- | -- | |
| | AII (symplectic) | -1 | 0 | 0 | -- | $Z_2$ | $Z_2$ | QSHE; $Z_2$TPI |
| Chiral | AIII (chiral unitary) | 0 | 0 | 1 | Z | -- | Z | |
| | BDI (chiral orthogonal) | +1 | +1 | 1 | Z | -- | -- | polyacetylene (SSH) |
| | CII (chiral symplectic) | -1 | -1 | 1 | Z | -- | $Z_2$ | |
| BdG; Majorana → | D (p-wave SC) | 0 | +1 | 0 | $Z_2$ | Z | -- | p SC; p+ip SC |
| | C (d-wave SC) | 0 | -1 | 0 | -- | Z | -- | d+id SC |
| Majorana → | DIII (p-wave TRS SC) | -1 | +1 | 1 | $Z_2$ | $Z_2$ | Z | (p+ip)x(p-ip) SC; $^3$He-B |
| | CI (d-wave TRS SC) | +1 | -1 | 1 | -- | -- | Z | |

Altland & Zirnbauer, PRB (1997)

Schnyder, Ryu, AF, and Ludwig, PRB (2008)

# Periodic table of topological insulators/superconductors

| | $d$ | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Cartan | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | … |
| *Complex case:* | | | | | | | | | | | | | |
| A | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | |
| AIII | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | |
| *Real case:* | | | | | | | | | | | | | |
| AI | $\mathbb{Z}$ | 0 | 0 | 0 | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | 0 | … |
| BDI | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | 0 | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | … |
| D | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | 0 | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | |
| DIII | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | 0 | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | |
| AII | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | 0 | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | |
| CII | 0 | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | 0 | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | … |
| C | 0 | 0 | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | 0 | $2\mathbb{Z}$ | 0 | … |
| CI | 0 | 0 | 0 | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | 0 | $2\mathbb{Z}$ | … |

period $d = 2$

period $d = 8$

A. Kitaev, AIP Conf. Proc. 1134, 22 (2009); arXiv:0901.2686 K-theory, Bott periodicity

Ryu, Schnyder, AF, Ludwig, NJP 12, 065010 (2010) massive Dirac Hamiltonian

M. Stone, C.-K. Chiu, A. Roy, J. Phys. A 44, 045001 (2011) representation of Clifford algebras

# Table of topological insulators/superconductors for d=1,2,3

| | 10 Symmetry Classes | TRS | PHS | CS | d=1 | d=2 | d=3 |
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| Standard (Wigner-Dyson) | A (unitary) | 0 | 0 | 0 | -- | Z | -- |
| | AI (orthogonal) | +1 | 0 | 0 | -- | -- | -- |
| | AII (symplectic) | -1 | 0 | 0 | -- | $Z_2$ | $Z_2$ |
| Chiral | AIII (chiral unitary) | 0 | 0 | 1 | Z | -- | Z |
| | BDI (chiral orthogonal) | +1 | +1 | 1 | Z | -- | -- |
| | CII (chiral symplectic) | -1 | -1 | 1 | Z | -- | $Z_2$ |
| BdG | D (p-wave SC) | 0 | +1 | 0 | $Z_2$ | Z | -- |
| | C (d-wave SC) | 0 | -1 | 0 | -- | Z | -- |
| | DIII (p-wave TRS SC) | -1 | +1 | 1 | $Z_2$ | $Z_2$ | Z |
| | CI (d-wave TRS SC) | +1 | -1 | 1 | -- | -- | Z |

Altland & Zirnbauer, PRB (1997)

Schnyder, Ryu, AF, and Ludwig, PRB (2008)

# Time-reversal operator

$$H = \sum_{i,j} c_i^\dagger H_{ij} c_j$$

Spin 0 case $T = \mathrm{K}$ $T:\ H_{ij} \rightarrow T H_{ij} T^{-1} = H_{ij}^*$

Complex conjugation $T^2 = 1$ integer Spin

Spin ½ case $T = i\sigma_y \mathrm{K}$ $T:\ H_{ij} \rightarrow T H_{ij} T^{-1} = \sigma_y H_{ij}^* \sigma_y$

$T^2 = -1$

Time-reversal invariant system:

$$T H_{ij} T^{-1} = H_{ij}$$

$H_{-\vec{k}}^* = H_{\vec{k}}$ Spin 0

$\sigma_y H_{-\vec{k}}^* \sigma_y = H_{\vec{k}}$ Spin ½

Example: 2D Dirac Hamiltonian

$$H(\vec{k}) = k_x\sigma_x + k_y\sigma_y + m\sigma_z + V\sigma_0$$

$$\sigma_y H^*(-\vec{k})\sigma_y = k_x\sigma_x + k_y\sigma_y - m\sigma_z + V\sigma_0$$

If $m = 0$, $H$ is invariant under time-reversal transformation $T$ $(T^2 = -1)$

Dirac fermion on the surface of a 3D $Z_2$ topological insulator

The mass term breaks time-reversal symmetry;

⟶ Quantum anomalous Hall effect $\sigma_{xy} = -\frac{e^2}{2h}\text{sgn}(m)$

# Classification of Hamiltonian in terms of time-reversal symmetry

$$\text{TRS} = \begin{cases} +1 & \text{if } THT^{-1} = H \text{ and } T^2 = +1 \\ -1 & \text{if } THT^{-1} = H \text{ and } T^2 = -1 \\ 0 & \text{if no } T \text{ exists.} \end{cases}$$

# Table of topological insulators/superconductors

| | | TRS | PHS | CS | d=1 | d=2 | d=3 |
|---|---|---|---|---|---|---|---|
| Standard (Wigner-Dyson) | A (unitary) | 0 | 0 | 0 | -- | Z | -- |
| | AI (orthogonal) | +1 | 0 | 0 | -- | -- | -- |
| | AII (symplectic) | -1 | 0 | 0 | -- | $Z_2$ | $Z_2$ |
| Chiral | AIII (chiral unitary) | 0 | 0 | 1 | Z | -- | Z |
| | BDI (chiral orthogonal) | +1 | +1 | 1 | Z | -- | -- |
| | CII (chiral symplectic) | -1 | -1 | 1 | Z | -- | $Z_2$ |
| BdG | D (p-wave SC) | 0 | +1 | 0 | $Z_2$ | Z | -- |
| | C (d-wave SC) | 0 | -1 | 0 | -- | Z | -- |
| | DIII (p-wave TRS SC) | -1 | +1 | 1 | $Z_2$ | $Z_2$ | Z |
| | CI (d-wave TRS SC) | +1 | -1 | 1 | -- | -- | Z |

# Particle-hole transformation for Bogoliubov-de Gennes Hamiltonian

Examples:

## (1) spinless $p_x+ip_y$

$$H=\frac{1}{2}\sum_{\vec{k}}\begin{pmatrix} c_{\vec{k}}^{\dagger} & c_{-\vec{k}} \end{pmatrix} H_{\vec{k}} \begin{pmatrix} c_{\vec{k}} \\ c_{-\vec{k}}^{\dagger} \end{pmatrix}$$

$$H_{\vec{k}}=\begin{pmatrix} \varepsilon_{\vec{k}} & \Delta\left(k_x-ik_y\right) \\ \Delta\left(k_x+ik_y\right) & -\varepsilon_{-\vec{k}} \end{pmatrix}=\Delta\left(k_x\tau_x+k_y\tau_y\right)+\varepsilon_k\tau_z$$

Particle-hole symmetry $\tau_x H_{-\vec{k}}^{*}\tau_x=-H_{\vec{k}}$ $\quad C=\tau_x \mathrm{K}$ $\quad C^2=1$

$$E_n\to -E_n$$

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix}\to\begin{pmatrix} v_n^{*} \\ u_n^{*} \end{pmatrix}$$

$$\begin{pmatrix} \psi \\ \psi^{\dagger} \end{pmatrix}=\sum_{E_n>0}\left[\begin{pmatrix} u_n \\ v_n \end{pmatrix}a_n+\begin{pmatrix} v_n^{*} \\ u_n^{*} \end{pmatrix}a_n^{\dagger}\right]+\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}\gamma_0$$

$$\gamma_0=\gamma_0^{\dagger}$$

Majorana fermion

$$u_0=v_0^{*}$$

# Particle-hole transformation for Bogoliubov-de Gennes Hamiltonian

(2) $d_{x2-y2}+id_{xy}$ (spin singlet pairing)

$$H = \sum_{\vec{k}} \begin{pmatrix} c_{k\uparrow}^{\dagger} & c_{-k\downarrow} \end{pmatrix} H_k \begin{pmatrix} c_{k\uparrow} \\ c_{-k\downarrow}^{\dagger} \end{pmatrix}$$

$$H_{\vec{k}} = \begin{pmatrix} \varepsilon_{\vec{k}} & \Delta\left(k_x^2 - k_y^2 - ik_x k_y\right) \\ \Delta\left(k_x^2 - k_y^2 + ik_x k_y\right) & -\varepsilon_{-\vec{k}} \end{pmatrix}$$

$$= \Delta\left[\left(k_x^2 - k_y^2\right)\tau_x + k_x k_y \tau_y\right] + \varepsilon_k \tau_z$$

Particle-hole symmetry $\tau_y H_{-\vec{k}}^{*} \tau_y = -H_{\vec{k}}$ $C = i\tau_y \mathrm{K}$ $C^2 = -1$

$$E_n \to -E_n$$

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} \to \begin{pmatrix} v_n^{*} \\ -u_n^{*} \end{pmatrix}$$

$$\begin{pmatrix} \psi_{\uparrow} \\ \psi_{\downarrow}^{\dagger} \end{pmatrix} = \sum_{E_n>0} \left[ \begin{pmatrix} u_n \\ v_n \end{pmatrix} a_{n\uparrow} + \begin{pmatrix} v_n^{*} \\ -u_n^{*} \end{pmatrix} a_{n\downarrow}^{\dagger} \right]$$

No Majorana

# Classification of Hamiltonian in terms of particle-hole symmetry

$$\text{PHS} = \begin{cases} +1 & \text{if } C^{-1}HC = -H \text{ and } C^2 = +1 \\ -1 & \text{if } C^{-1}HC = -H \text{ and } C^2 = -1 \\ 0 & \text{if no } C \text{ exists.} \end{cases}$$

# Table of topological insulators/superconductors

| | | TRS | PHS | CS | d=1 | d=2 | d=3 |
|---|---|---|---|---|---|---|---|
| Standard (Wigner-Dyson) | A (unitary) | 0 | 0 | 0 | -- | Z | -- |
| | AI (orthogonal) | +1 | 0 | 0 | -- | -- | -- |
| | AII (symplectic) | -1 | 0 | 0 | -- | $Z_2$ | $Z_2$ |
| Chiral | AIII (chiral unitary) | 0 | 0 | 1 | Z | -- | Z |
| | BDI (chiral orthogonal) | +1 | +1 | 1 | Z | -- | -- |
| | CII (chiral symplectic) | -1 | -1 | 1 | Z | -- | $Z_2$ |
| BdG | D (p-wave SC) | 0 | +1 | 0 | $Z_2$ | Z | -- |
| | C (d-wave SC) | 0 | -1 | 0 | -- | Z | -- |
| | DIII (p-wave TRS SC) | -1 | +1 | 1 | $Z_2$ | $Z_2$ | Z |
| | CI (d-wave TRS SC) | +1 | -1 | 1 | -- | -- | Z |

# Chiral symmetry (CS)

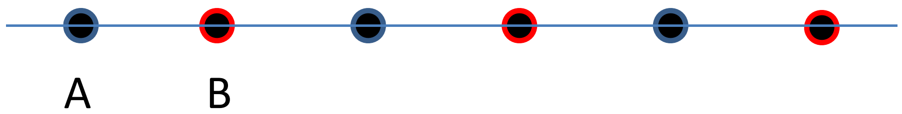
There is a unitary operator which anticommutes with Hamiltonian.

$$H\Gamma + \Gamma H = 0$$

$$H = \begin{pmatrix} 0 & D \\ D^{\dagger} & 0 \end{pmatrix} \qquad \Gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Example 1: lattice model with hopping between AB sublattices only

$$H = \sum_{\substack{a\in A \\ b\in B}} \left( t_{ab} c_a^{\dagger} c_b + t_{ab}^{*} c_b^{?} c_a \right)$$




Example 2: time-reversal × particle-hole ($T$ and $C$ are antiunitary)

$$THT^{-1} = H$$

$$CHC^{-1} = -H$$

$$\Longrightarrow \quad TCHC^{-1}T^{-1} = -H \qquad TCH = -HTC$$

# Classification of free-fermion Hamiltonian in terms of generic discrete symmetries

- Time-reversal symmetry (TRS)

$$THT^{-1} = H$$

$$\mathrm{TRS} = \begin{cases} 0 & \text{no TR invariance} \\ +1 & T^2 = +1 \\ -1 & T^2 = -1 \end{cases}$$

spin 0

spin 1/2

- Particle-hole symmetry (PHS)

BdG Hamiltonian

$$CHC^{-1} = -H$$

$$\mathrm{PHS} = \begin{cases} 0 & \text{no PH invariance} \\ +1 & C^2 = +1 \\ -1 & C^2 = -1 \end{cases}$$

triplet

singlet

$$\mathrm{TRS} = \mathrm{PHS} = 0,\ \mathrm{CS} = 1$$

$$3 \times 3 + 1 = 10$$

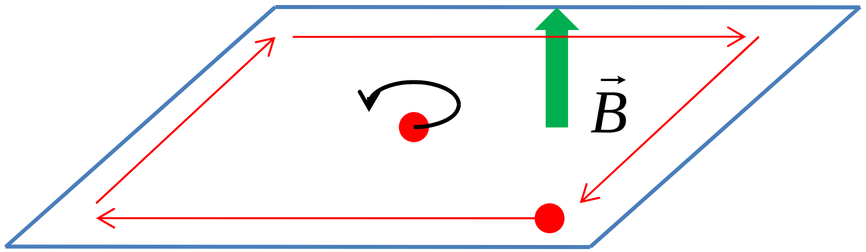
# Table of topological insulators/superconductors for d=1,2,3

| | | TRS | PHS | CS | d=1 | d=2 | d=3 |
|---|---|---|---|---|---|---|---|
| Standard (Wigner-Dyson) | A (unitary) | 0 | 0 | 0 | -- | Z (IQHE) | -- |
| | AI (orthogonal) | +1 | 0 | 0 | -- | -- | -- |
| | AII (symplectic) | -1 | 0 | 0 | -- | $Z_2$ (QSHE) | $Z_2$ ($Z_2$TPI) |
| Chiral | AIII (chiral unitary) | 0 | 0 | 1 | Z | -- | Z |
| | BDI (chiral orthogonal) | +1 | +1 | 1 | Z | -- | -- |
| | CII (chiral symplectic) | -1 | -1 | 1 | Z | -- | $Z_2$ |
| BdG | Majorana → D (p-wave SC) | 0 | +1 | 0 | $Z_2$ | Z (p+ip SC) | -- |
| | C (d-wave SC) | 0 | -1 | 0 | -- | Z | -- |
| | Majorana → DIII (p-wave TRS SC) | -1 | +1 | 1 | $Z_2$ | $Z_2$ ((p+ip)x(p-ip) SC) | Z ($^3$He-B) |
| | CI (d-wave TRS SC) | +1 | -1 | 1 | -- | -- | Z |

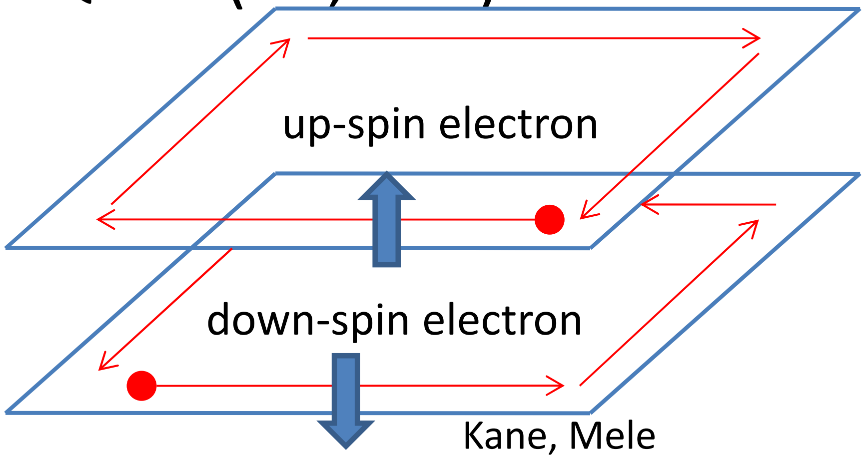
Schnyder, Ryu, AF, and Ludwig, PRB (2008)

## IQHE (A, d=2)

$$H(\vec{k}) = k_x\sigma_x + k_y\sigma_y + m\sigma_z + V\sigma_0$$

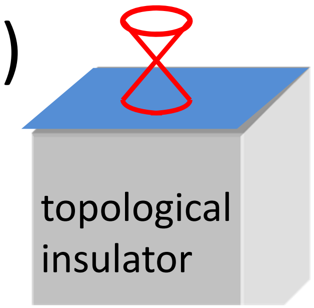



## QSHE (AII, d=2)




Kane, Mele
Bernevig, Hughes, Zhang

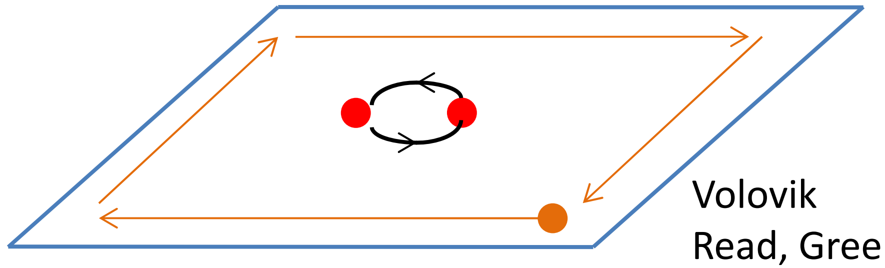
## 3D $Z_2$ TI (AII, d=3)




Moore, Balents
Fu, Kane, Mele
Roy

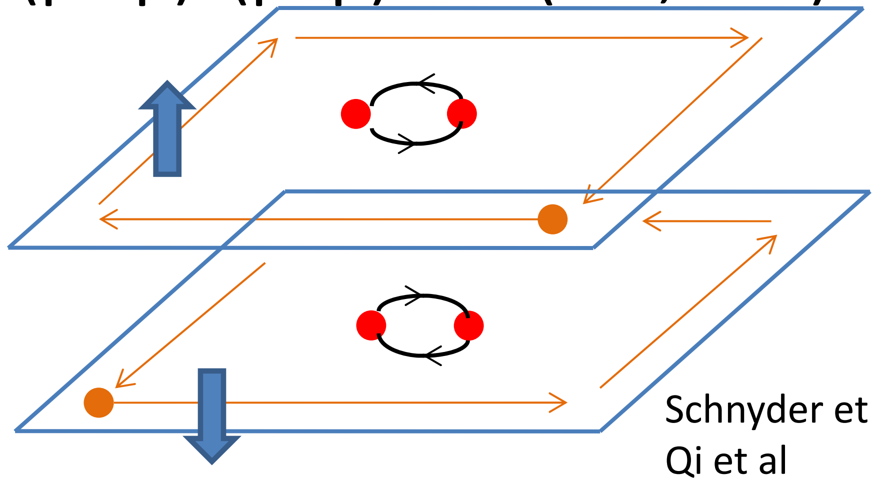
## (p+ip) SC (D, d=2)

$$H(\vec{k}) = \Delta\left(k_x\tau_x + k_y\tau_y\right) + \varepsilon_k\tau_z$$



Volovik
Read, Green

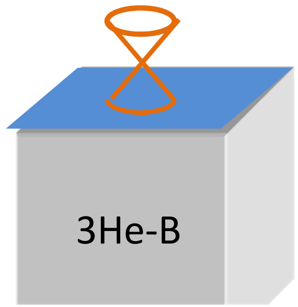
## (p+ip)x(p-ip) SC (DIII, d=2)



Schnyder et al
Qi et al
Sato, Fujimoto

NCS SC (Rashba SO; p > s)

## $^3$He-B (DIII, d=3)




Volovik, ....
Schnyder et al

# “derivation” of the periodic table

- Anderson delocalization of boundary states
  - Nonlinear sigma model with a topological term
- Dirac Hamiltonian
  - dimensional reduction (complex classes)
  - Clifford algebras

# Anderson delocalization of boundary states

- Gapless boundary modes are topologically protected.
- They are stable against any local perturbation.
  (respecting discrete symmetries)
- They should never be Anderson localized by disorder.

Nonlinear sigma models for Anderson localization
of gapless boundary modes

$$S = \int d^{d-1}r \, \mathrm{tr}\left(\partial Q\right)^2 + \text{topological term}$$ (with no adjustable parameter)

$Q \in M$

$Z_2$ top. term $\quad \pi_{d-1}(M) = Z_2$

WZW term $\quad \pi_d(M) = Z$

~~$\theta$-term~~

bulk: $d$ dimensions
boundary: $d$ -1 dimensions

# NLSM topological terms $\pi_d(G/H)$

*complex case:*

| | $G/H \setminus d$ | $d=0$ | $d=1$ | $d=2$ | $d=3$ |
|---|---|---|---|---|---|
| A | $U(N+M)/U(N)\times U(M)$ | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 |
| AIII | $U(N)$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ |

*real case:*

| | $G/H \setminus d$ | $d=0$ | $d=1$ | $d=2$ | $d=3$ |
|---|---|---|---|---|---|
| AI | $Sp(N+M)/Sp(N)\times Sp(M)$ | $\mathbb{Z}$ | 0 | 0 | 0 |
| BDI | $U(2N)/Sp(N)$ | 0 | $\mathbb{Z}$ | 0 | 0 |
| D | $O(2N)/U(N)$ | $\mathbb{Z}_2$ | 0 | $\mathbb{Z}$ | 0 |
| DIII | $O(N)$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | 0 | $\mathbb{Z}$ |
| AII | $O(N+M)/O(N)\times O(M)$ | $\mathbb{Z}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | 0 |
| CII | $U(N)/O(N)$ | 0 | $\mathbb{Z}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ |
| C | $Sp(N)/U(N)$ | 0 | 0 | $\mathbb{Z}$ | $\mathbb{Z}_2$ |
| CI | $Sp(N)$ | 0 | 0 | 0 | $\mathbb{Z}$ |

$Z_2$: $Z_2$ topological term can exist in d dimensions ⟹ d+1 dim. TI/TSC

Z: WZW term can exist in d-1 dimensions ⟹ d dim. TI/TSC

# Periodic table of topological insulators/superconductors

| Cartan | $d$ = 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | … |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| *Complex case:* | | | | | | | | | | | | | |
| A | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | |
| AIII | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | |
| *Real case:* | | | | | | | | | | | | | |
| AI | $\mathbb{Z}$ | 0 | 0 | 0 | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | 0 | … |
| BDI | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | 0 | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | … |
| D | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | 0 | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | |
| DIII | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | 0 | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | |
| AII | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | 0 | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | |
| CII | 0 | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | 0 | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | … |
| C | 0 | 0 | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | 0 | $2\mathbb{Z}$ | 0 | … |
| CI | 0 | 0 | 0 | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | 0 | $2\mathbb{Z}$ | … |

period $d = 2$

period $d = 8$

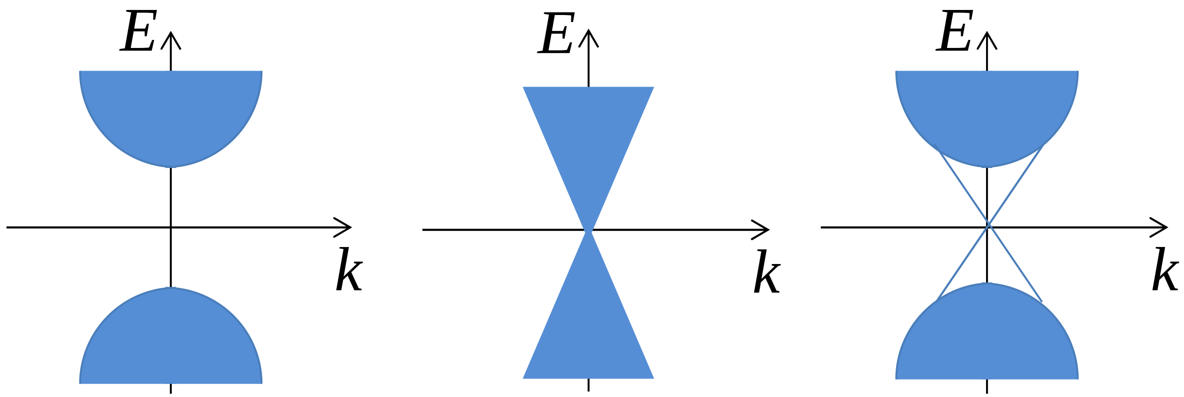
A. Kitaev, AIP Conf. Proc. 1134, 22 (2009); arXiv:0901.2686    K-theory, Bott periodicity

Ryu, Schnyder, AF, Ludwig, NJP 12, 065010 (2010)    massive Dirac Hamiltonian

# Dirac Hamiltonian

Minimal representative models for TIs and TSCs

Effective theory near a topological phase transition (band gap closing)




$$H = \sum_{\mu=1}^{d} k_\mu \gamma_\mu + m\gamma_{d+1}$$

gamma matrices:

$$\{\gamma_a, \gamma_b\} = 2\delta_{a,b}$$

$2^n \times 2^n$ matrices

$$\Gamma^1_{(2n+1)} = \sigma_1 \otimes \underbrace{\sigma_3 \otimes \cdots \otimes \sigma_3}_{n-1},$$

$$\Gamma^2_{(2n+1)} = \sigma_2 \otimes \underbrace{\sigma_3 \otimes \cdots \otimes \sigma_3}_{n-1},$$

$$\Gamma^3_{(2n+1)} = \sigma_0 \otimes \sigma_1 \otimes \underbrace{\sigma_3 \otimes \cdots \otimes \sigma_3}_{n-2},$$

$$\Gamma^4_{(2n+1)} = \sigma_0 \otimes \sigma_2 \otimes \underbrace{\sigma_3 \otimes \cdots \otimes \sigma_3}_{n-2},$$

$$\vdots$$

$$\Gamma^{2n-1}_{(2n+1)} = \underbrace{\sigma_0 \otimes \cdots \otimes \sigma_0}_{n-1} \otimes \sigma_1,$$

$$\Gamma^{2n}_{(2n+1)} = \underbrace{\sigma_0 \otimes \cdots \otimes \sigma_0}_{n-1} \otimes \sigma_2,$$

$$\Gamma^{2n+1}_{(2n+1)} = \underbrace{\sigma_3 \otimes \cdots \otimes \sigma_3}_{n}.$$

# Dimensional hierarchy: complex case (A ➡ AIII)

d=2n: class A (no symmetry constraint; e.g., IQHE)

$$H = \sum_{\mu=1}^{2n} k_\mu \gamma_\mu + m\gamma_{2n+1} \qquad \gamma_1,\ \ldots,\ \gamma_{2n+1}$$

Bloch wave functions of occupied bands $\left|u_a\left(\vec{k}\right)\right\rangle \quad a = 1, \ldots, N$

Berry connection $A_\mu^{ab}\left(\vec{k}\right) dk_\mu = \left\langle u_a\left(\vec{k}\right)\middle| du_b\left(\vec{k}\right)\right\rangle$

Berry curvature $F = dA + A \wedge A$

Chern number $\mathrm{Ch}_n\left[F\right] = \int \frac{1}{(n+1)!} \mathrm{tr}\left(\frac{iF}{2\pi}\right)^n \quad \in \mathbf{Z}$

d=2n-1: class AIII

$$H = \sum_{\mu=1}^{2n-1} k_\mu \gamma_\mu + m\gamma_{2n+1}$$

$$\{H, \gamma_{2n}\} = 0$$ chiral symmetry ⇒ $$H(\vec{k}) = \begin{pmatrix} 0 & D(\vec{k}) \\ D^\dagger(\vec{k}) & 0 \end{pmatrix}$$

Deform the Hamiltonian continuously to a Hamiltonian with eigenvalues $\pm 1$

$$Q(\vec{k}) = 1 - 2\sum_{a=1}^{N} \left| u_a(\vec{k}) \right\rangle \left\langle u_a(\vec{k}) \right| = \begin{pmatrix} 0 & q(\vec{k}) \\ q^\dagger(\vec{k}) & 0 \end{pmatrix} \qquad q(\vec{k}) \in \mathrm{U(N)}$$

$$\nu_{2n-1}[q] = \int d^{2n-1}k \, \frac{i^n (-1)^{n-1} (n-1)!}{(2\pi)^n (2n-1)!} \varepsilon^{\alpha_1 \alpha_2 \cdots \alpha_{2n-1}} \mathrm{tr}\left[ \left(q^{-1}\partial_{\alpha_1} q\right)\left(q^{-1}\partial_{\alpha_2} q\right)\cdots\left(q^{-1}\partial_{\alpha_{2n-1}} q\right) \right] \in \mathbf{Z}$$

$$\pi_{2n-1}\left(\mathrm{U(N)}\right) = \mathbf{Z}$$

# Example: $d = 3 \rightarrow 2 \rightarrow 1$

$$\Gamma_3^{a=1,2,3} = \{\sigma_x, \sigma_y, \sigma_z\}$$

$d = 3$ $\quad H = k_x\sigma_x + k_y\sigma_y + k_z\sigma_z$ Weyl semimetal

$d = 2$ $\quad H = k_x\sigma_x + k_y\sigma_y + m\sigma_z$ Class A (IQHE)

$$\mathrm{Ch}_1 = \frac{i}{2\pi}\int d^2k F_{xy} = \frac{i}{2\pi}\int d^2k \frac{-im}{2\left(k^2+m^2\right)^{3/2}} = \frac{m}{2|m|} = \sigma_{xy}$$

$d = 1$ $\quad H = k_x\sigma_x + m\sigma_y$

$$q(k) = -\frac{k_x + im}{\sqrt{k_x^2 + m^2}}$$

$$\nu_1 = \frac{i}{2\pi}\int q^{-1}dq = \frac{i}{2\pi}\int dk_x \frac{-im}{k_x^2 + m^2} = \frac{m}{2|m|}$$

# Classification of Dirac mass

$$H = \sum_{\mu=1}^{d} k_\mu \gamma_\mu + m\gamma_0 \qquad \{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu,\nu}$$

If $m\gamma_0$ is a unique Dirac mass, then gapped phases with opposite sign of $m$ are topologically distinct phases.

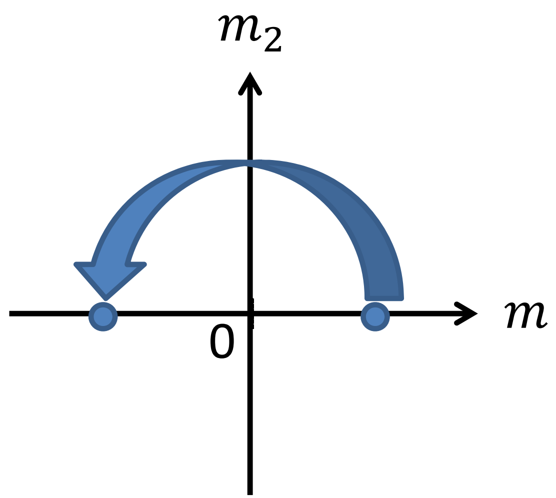
Examples

(1) $d = 2$ class A (IQHE)

$$H = k_x\sigma_x + k_y\sigma_y + m\sigma_z$$

(2) $d = 1$ class A

$$H = k_x\sigma_x + m_1\sigma_y + m_2\sigma_z$$




Two gapped states ($m_1 > 0$ and $m_1 < 0$) are connected without closing a gap.

(3) $d = 1$ class AIII $\quad \{H, \sigma_z\} = 0$

$$H = k_x\sigma_x + m\sigma_y$$

$m\sigma_y$ is a unique mass term.

# Set of possible mass terms: classifying space

Example: $d = 2$ class A (IQHE)

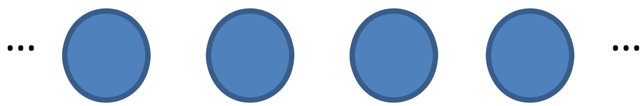
$$H = k_x \underbrace{\sigma_x \otimes 1_N}_{\gamma_1} + k_y \underbrace{\sigma_y \otimes 1_N}_{\gamma_2} + \gamma_0 \qquad \{\gamma_a, \gamma_b\} = 2\delta_{a,b}$$

$$\gamma_0 = \sigma_z \otimes A \qquad A = U \begin{pmatrix} 1_n & 0 \\ 0 & -1_m \end{pmatrix} U^\dagger \qquad (N = n + m)$$

$$\gamma_0 \Longleftrightarrow U \in \frac{U(n+m)}{U(n) \times U(m)}$$

Classifying space $C_0$
= Complex Grassmanian

$$\pi_0 \left[ \oplus_{m,n} U(m+n) / U(m) \times U(n) \right] = \mathbf{Z}$$



There are topologically distinct gapped phases labelled by an integer index.

The parameter $n$ corresponds to Chern number.

$$H = k_x \sigma_x + k_y \sigma_y + \left(\varepsilon - k^2\right) \sigma_z \qquad \text{Chern \# } = \begin{cases} 1 & (\varepsilon > 0) \\ 0 & (\varepsilon < 0) \end{cases}$$

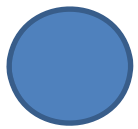
Example: $d = 1$ class A (no symmetry constraint)

$$H = k_x \sigma_z \otimes 1_N + \gamma_0$$

$$\gamma_0 = \begin{pmatrix} 0 & U \\ U^\dagger & 0 \end{pmatrix} \qquad U \in U(N) \qquad \text{Classifying space } C_1$$

$$\pi_0\left(U(N)\right) = 0$$

There is only a single gapped phase.



# Classification using Clifford algebras (real classes)

(real) Clifford algebra $Cl_{p,q}$

$p + q$ generators:

$$\{e_i, e_j\} = 0 \quad (i \neq j)$$

$$e_i^2 = \begin{cases} -1 & (i = 1, \ldots, p) \\ +1 & (i = p+1, \ldots, p+q) \end{cases}$$

$2^{p+q}$-dimensional real vector space

$$a_1 e_1 + a_2 e_2 + \ldots + a_{12} e_1 e_2 + \ldots + a_{12\ldots n} e_1 e_2 \cdots e_n \qquad a_i \in \mathbb{R}$$

# Symmetry operators = generators of Clifford algebras

Time-reversal transformation: $T$ $\quad T^{-1}HT = H, \qquad T^2 = \pm 1$

Particle-hole transformation: $C$ $\quad C^{-1}HC = -H, \qquad C^2 = \pm 1$

$$\left[T, C\right] = 0$$

Operator for "$i$" : $J$ $\qquad J^2 = -1, \quad \left\{T, J\right\} = \left\{C, J\right\} = \left[H, J\right] = 0$

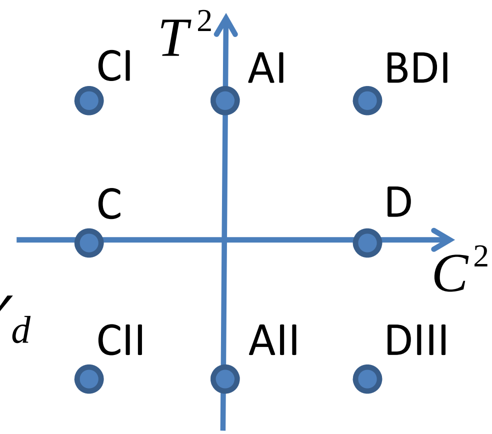
Dirac Hamiltonian $H = \sum_{\mu=1}^{d} k_\mu \gamma_\mu + m\gamma_0$

$$T\gamma_0 = \gamma_0 T, \quad T\gamma_\mu = -\gamma_\mu T \quad (\mu = 1, \ldots, d)$$

$$C\gamma_0 = -\gamma_0 C, \quad C\gamma_\mu = \gamma_\mu C \quad (\mu = 1, \ldots, d)$$

$$\left\{\gamma_a, \gamma_b\right\} = 2\delta_{a,b}$$

# Real Clifford algebras




(i) $T$ only (AI & AII):

$$e_0 = J\gamma_0, \quad e_1 = T, \quad e_2 = TJ, \quad e_3 = \gamma_1, \quad ...., \quad e_{2+d} = \gamma_d$$

AI: $Cl_{1,2+d}$ AII: $Cl_{3,d}$

(ii) $C$ only (C & D):

$$e_0 = \gamma_0, \quad e_1 = C, \quad e_2 = CJ, \quad e_3 = J\gamma_1, \quad ...., \quad e_{2+d} = J\gamma_d$$

C: $Cl_{2+d,1}$ D: $Cl_{d,3}$

(iii) $T$ and $C$ (BDI, DIII, CII & CI):

$$e_0 = \gamma_0, \quad e_1 = C, \quad e_2 = CJ, \quad e_3 = TCJ, \quad e_4 = J\gamma_1, \quad ...., \quad e_{3+d} = J\gamma_d$$

BDI: $Cl_{1+d,3}$ DIII: $Cl_{d,4}$ CII: $Cl_{3+d,1}$ CI: $Cl_{2+d,2}$

# Topological classification of Hamiltonian (Kitaev 2009)

(1) We consider a matrix representation (of large enough dimension) of a Clifford algebra without $e_0$.
(We fix the representation for the symmetry constraints.)

(2) We then consider extending Clifford algebras by adding $e_0$.

(i), (ii) $\{e_1, e_2, \ldots, e_{2+d}\} \to \{e_0, e_1, e_2, \ldots, e_{2+d}\}$

(iii) $\{e_1, e_2, \ldots, e_{2+d}\} \to \{e_0, e_1, e_2, \ldots, e_{2+d}\}$

We look for all possible representations of $e_0$.

The set of possible $e_0$ : classifying space $R_q$ $(q = 0,1, \ldots, 7)$

The classifying space for $\{e_1, e_2, \ldots, e_{2+d}\} \to \{e_0, e_1, e_2, \ldots, e_{2+d}\}$

$\{e_1, e_2, \ldots, e_{2+d}\} \to \{e_0, e_1, e_2, \ldots, e_{2+d}\}$

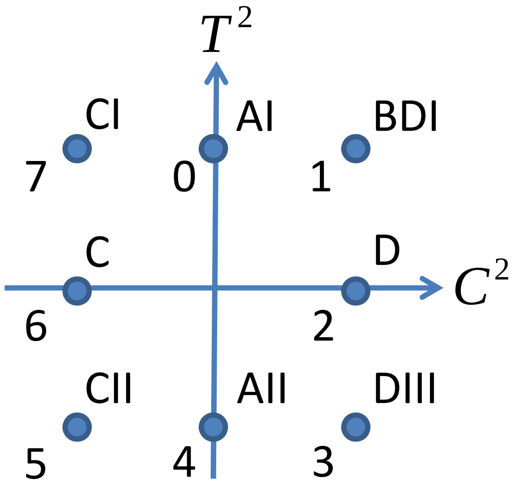
(3) Topological classification is given by $\pi_0\left(R_q\right)$.

Bott periodicity $R_{q+8} = R_q$

# Classification of TIs and TSCs in $d = 0$

| class | $(T^2, C^2)$ | extension | classifying space |
|---|---|---|---|
| AI | $(+, 0)$ | $Cl_{0,2} \to Cl_{1,2}$ | $R_0$ |
| AII | $(-, 0)$ | $Cl_{2,0} \to Cl_{3,0}$ | $R_4$ |
| D | $(0, +)$ | $Cl_{0,2} \to Cl_{0,3}$ | $R_2$ |
| C | $(0, -)$ | $Cl_{2,0} \to Cl_{2,1}$ | $R_{-2} \simeq R_6$ |
| BDI | $(+, +)$ | $Cl_{1,2} \to Cl_{1,3}$ | $R_1$ |
| DIII | $(-, +)$ | $Cl_{0,3} \to Cl_{0,4}$ | $R_3$ |
| CII | $(-, -)$ | $Cl_{3,0} \to Cl_{3,1}$ | $R_{-3} \simeq R_5$ |
| CI | $(+, -)$ | $Cl_{2,1} \to Cl_{2,2}$ | $R_{-1} \simeq R_7$ |

| class | TRS | PHS | $R_q$ | $\pi_0(R_q)$ |
|---|---|---|---|---|
| AI | $+1$ | 0 | $R_0$ | $\mathbb{Z}$ |
| BDI | $+1$ | $+1$ | $R_1$ | $\mathbb{Z}_2$ |
| D | 0 | $+1$ | $R_2$ | $\mathbb{Z}_2$ |
| DIII | $-1$ | $+1$ | $R_3$ | 0 |
| AII | $-1$ | 0 | $R_4$ | $\mathbb{Z}$ |
| CII | $-1$ | $-1$ | $R_5$ | 0 |
| C | 0 | $-1$ | $R_6$ | 0 |
| CI | $+1$ | $-1$ | $R_7$ | 0 |




0 dimension $R_q$

$d$ dimensions $R_{q-d}$

# Dirac Hamiltonians in $d$ dimensions

$$H = \sum_{\mu=1}^{d} k_{\mu}\gamma_{\mu} + m\gamma_{0}$$

The relevant classifying space is $R_{q-d}$.

Topological classification is found from $\pi_0(R_{q-d})$.

Bott periodicity $R_{q+8} \cong R_q$

| | $d$ | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Cartan | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | … |
| *Complex case:* | | | | | | | | | | | | | |
| A | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | … |
| AIII | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | … |
| *Real case:* | | | | | | | | | | | | | |
| AI | $\mathbb{Z}$ | 0 | 0 | 0 | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | 0 | … |
| BDI | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | 0 | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | … |
| D | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | 0 | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | … |
| DIII | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | 0 | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | … |
| AII | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | 0 | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | … |
| CII | 0 | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | 0 | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | … |
| C | 0 | 0 | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | 0 | $2\mathbb{Z}$ | 0 | … |
| CI | 0 | 0 | 0 | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | 0 | $2\mathbb{Z}$ | … |

# Reflection symmetry

Dirac Hamiltonian $H = \sum_{\mu=1}^{d} k_\mu \gamma_\mu + m\gamma_0$

Reflection in the $x_1$ direction:

$$R^{-1} H(-k_1, k_i) R = H(k_1, k_i) \Rightarrow \{R, \gamma_1\} = 0, \quad [R, \gamma_i] = 0 \ \ (i \neq 1)$$

Define $M = J\gamma_1 R$ , which satisfies $M^2 = 1$ and $\{M, \gamma_\mu\} = 0$.

Suppose that $MT = \eta_T TM$ and/or $MC = \eta_C CM$ . $(\eta_{T/C} = +1 \text{ or } -1)$

$$\Updownarrow$$

$RT = \eta_T TR$ and/or $RC = -\eta_C CR$

$$R^{\eta_T},\ R^{-\eta_C},\ R^{\eta_T, -\eta_C}$$

The operator $M$ changes the relevant Clifford algebra.

$$MT = \eta_T TM \qquad MC = \eta_C CM$$

$$(\eta_T, \eta_C)$$

$\Downarrow$

(i) New generator $\tilde{e}$

→ Shift $R_q \to R_{q\pm1}$

| class | $(\eta_T, \eta_C)$ | $\tilde{e}$ | $\tilde{e}^2$ | shift of $R_q$ |
|---|---|---|---|---|
| AI, AII | $(+,0)$ | $JM$ | $-1$ | $+1$ |
| | $(-,0)$ | $M$ | $+1$ | $-1$ |
| D, C | $(0,+)$ | $JM$ | $-1$ | $-1$ |
| | $(0,-)$ | $M$ | $+1$ | $+1$ |
| BDI, DIII, CII, CI | $(+,-)$ | $M$ | $+1$ | $+1$ |
| | $(-,+)$ | $JM$ | $-1$ | $-1$ |

SnTe

(ii) Commuting operator $\widetilde{M}$

→ Block diagonalization

| class | $(\eta_T, \eta_C)$ | $\widetilde{M}$ | $\widetilde{M}^2$ | classifying space |
|---|---|---|---|---|
| BDI, CII | $(+,+)$ | $TCM$ | $+1$ | no change |
| | $(-,-)$ | $TCJM$ | $-1$ | complex |
| DIII, CI | $(+,+)$ | $TCM$ | $-1$ | no change |
| | $(-,-)$ | $TCJM$ | $+1$ | complex |

When $\widetilde{M}^2 = -1$, $\widetilde{M}$ introduces complex structure.

# Topological periodic table with a reflection symmetry

Original topological periodic table for ten AZ symmetry classes

| | $d$ | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Cartan | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | … |
| *Complex case:* | | | | | | | | | | | | | |
| A | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | period |
| AIII | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | d = 2 |
| *Real case:* | | | | | | | | | | | | | |
| AI | $\mathbb{Z}$ | 0 | 0 | 0 | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | 0 | … |
| BDI | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | 0 | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | … |
| D | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | 0 | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | |
| DIII | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | 0 | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | period |
| AII | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | 0 | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | d = 8 |
| CII | 0 | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | 0 | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | … |
| C | 0 | 0 | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | 0 | $2\mathbb{Z}$ | 0 | … |
| CI | 0 | 0 | 0 | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | 0 | $2\mathbb{Z}$ | … |

# Topological periodic table with a reflection symmetry

| Reflection | Class | $C_q$ or $R_q$ | $d=0$ | $d=1$ | $d=2$ | $d=3$ | $d=4$ | $d=5$ | $d=6$ | $d=7$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $R$ | A | $C_1$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ |
| $R^+$ | AIII | $C_0$ | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 |
| $R^-$ | AIII | $C_1$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ |
| $R^+, R^{++}$ | AI | $R_1$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ |
| | BDI | $R_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | 0 | $\mathbb{Z}$ | 0 |
| | D | $R_3$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | 0 | $\mathbb{Z}$ |
| | DIII | $R_4$ | $\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | 0 |
| | AII | $R_5$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 |
| | CII | $R_6$ | 0 | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 |
| | C | $R_7$ | 0 | 0 | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ |
| | CI | $R_0$ | $\mathbb{Z}$ | 0 | 0 | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ |
| $R^-, R^{--}$ | AI | $R_7$ | 0 | 0 | 0 | $\mathbb{Z}$ | 0 | "$\mathbb{Z}_2$" | $\mathbb{Z}_2$ | $\mathbb{Z}$ |
| | BDI | $R_0$ | $\mathbb{Z}$ | 0 | 0 | 0 | $\mathbb{Z}$ | 0 | "$\mathbb{Z}_2$" | $\mathbb{Z}_2$ |
| | D | $R_1$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | 0 | $\mathbb{Z}$ | 0 | "$\mathbb{Z}_2$" |
| | DIII | $R_2$ | "$\mathbb{Z}_2$" | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | 0 | $\mathbb{Z}$ | 0 |
| | AII | $R_3$ | 0 | "$\mathbb{Z}_2$" | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | 0 | $\mathbb{Z}$ |
| | CII | $R_4$ | $\mathbb{Z}$ | 0 | "$\mathbb{Z}_2$" | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | 0 |
| | C | $R_5$ | 0 | $\mathbb{Z}$ | 0 | "$\mathbb{Z}_2$" | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 |
| | CI | $R_6$ | 0 | 0 | $\mathbb{Z}$ | 0 | "$\mathbb{Z}_2$" | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 |
| $R^{+-}$ | BDI | $R_1$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ |
| $R^{-+}$ | DIII | $R_3$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | 0 | $\mathbb{Z}$ |
| $R^{+-}$ | CII | $R_5$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 |
| $R^{-+}$ | CI | $R_7$ | 0 | 0 | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ |
| $R^{-+}$ | BDI, CII | $C_1$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ |
| $R^{+-}$ | DIII, CI | $C_1$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ |

*Shifts*

SnTe

*Block diagonalization*

Chiu, Yao, & Ryu, PRB 88, 075142 (2013); Morimoto & AF, PRB 88, 125129 (2013).

# TCI SnTe as TRS + $R^{-}$

Hsieh et al. Nat. Commun. 2012

Band gaps at 4 L points

Effective theory around an L point

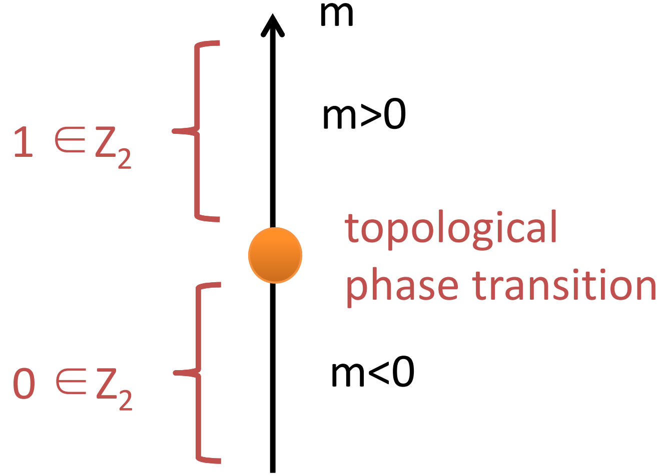
$$H = v\left(k_x s_y - k_y s_x\right)\sigma_x + v_z k_z \sigma_y + m\sigma_z$$

$\sigma_z = \pm 1$: p-orbitals, $s_z = \pm 1$: $j = \pm\frac{1}{2}$

class AII: $T = i s_y K$

unique mass term : $\sigma_z$

Topological index $Z_2$



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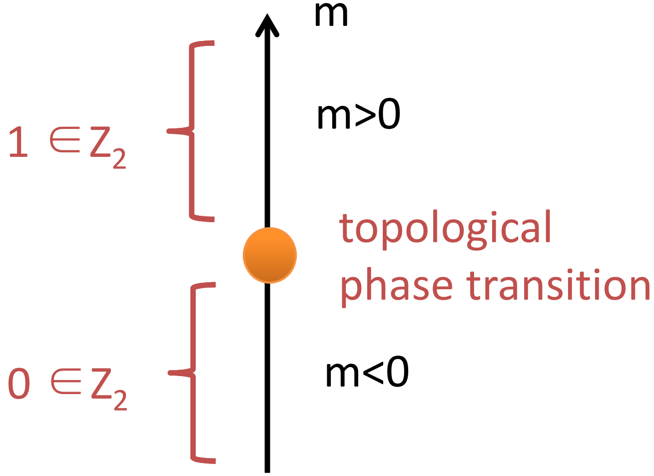
class AII: $T = is_y \mathrm{K}$

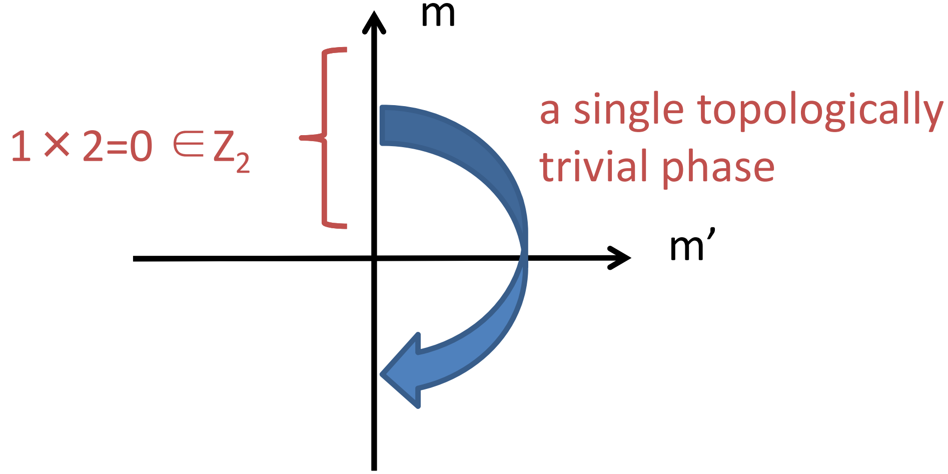
unique mass term : $\sigma_z$

Topological index $Z_2$

Doubled system $H \otimes \tau_0$

→ an extra mass term $m' s_z \sigma_x \tau_y$



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Hsieh et al. Nat. Commun. 2012

Band gaps at 4 L points

Effective theory around an L point

$$H = v\left(k_x s_y - k_y s_x\right)\sigma_x + v_z k_z \sigma_y + m\sigma_z$$

$\sigma_z = \pm 1$: p-orbitals, $s_z = \pm 1$: $j = \pm\frac{1}{2}$

class AII: $T = is_y K$

unique mass term : $\sigma_z$

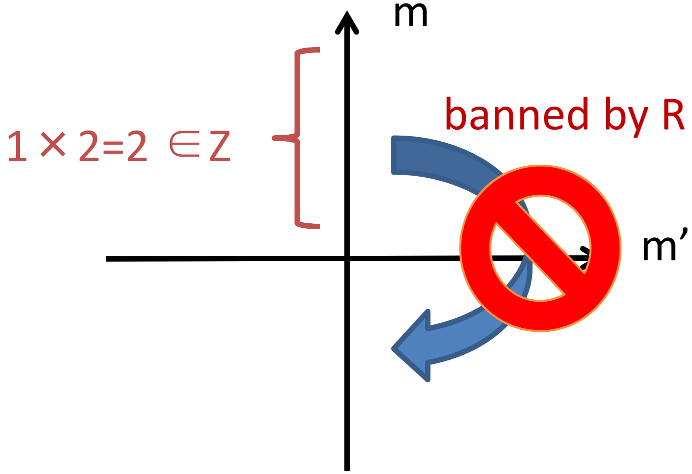
Topological index $Z_2$

Doubled system $H \otimes \tau_0$

→ an extra mass term $m' s_z \sigma_x \tau_y$

Reflection $R_x^{-} = s_x$ forbids the extra mass $m'$ → Topological index Z (mirror Chern number)

$(k_x \to -k_x)$




| class | TRS | PHS | $R_q$ | $\pi_0(R_q)$ |
|---|---|---|---|---|
| AI | +1 | 0 | $R_0$ | $\mathbb{Z}$ |
| BDI | +1 | +1 | $R_1$ | $\mathbb{Z}_2$ |
| D | 0 | +1 | $R_2$ | $\mathbb{Z}_2$ |
| DIII | −1 | +1 | $R_3$ | 0 |
| AII | −1 | 0 | $R_4$ | $\mathbb{Z}$ |
| CII | −1 | −1 | $R_5$ | 0 |
| C | 0 | −1 | $R_6$ | 0 |
| CI | +1 | −1 | $R_7$ | 0 |

$R^{-}$

$d = 3$

# Summary

- Periodic table of topological insulators/superconductors

  3 Z & 2 $Z_2$ in every dimension

  exhaustive list for any free fermion Hamiltonian
- Powerful machinery using Clifford algebras and their representations
- Generalizations (lattice symmetries other than reflections)
  - D.S. Freed & G.W. Moore, arXiv:1208.5055

- Weak points
  - Abstract toy models
  - Do not give topological invariants explicitly
  - Electronic correlations???

(a) complex classes

| $q$ | $Cl_q$ | $C_q$ | $\pi_0(C_q)$ |
|---|---|---|---|
| 0 | $\mathbb{C}$ | $(U(n+m)/U(n) \times U(m)) \times \mathbb{Z}$ | $\mathbb{Z}$ |
| 1 | $\mathbb{C} \oplus \mathbb{C}$ | $U(n)$ | 0 |

(b) real classes

| $q$ | $Cl_{0,q}$ | $R_q$ | $\pi_0(R_q)$ |
|---|---|---|---|
| 0 | $\mathbb{R}$ | $(O(n+m)/O(n) \times O(m)) \times \mathbb{Z}$ | $\mathbb{Z}$ |
| 1 | $\mathbb{R} \oplus \mathbb{R}$ | $O(n)$ | $\mathbb{Z}_2$ |
| 2 | $\mathbb{R}(2)$ | $O(2n)/U(n)$ | $\mathbb{Z}_2$ |
| 3 | $\mathbb{C}(2)$ | $U(2n)/Sp(n)$ | 0 |
| 4 | $\mathbb{H}(2)$ | $(Sp(n+m)/Sp(n) \times Sp(m)) \times \mathbb{Z}$ | $\mathbb{Z}$ |
| 5 | $\mathbb{H}(2) \oplus \mathbb{H}(2)$ | $Sp(n)$ | 0 |
| 6 | $\mathbb{H}(4)$ | $Sp(n)/U(n)$ | 0 |
| 7 | $\mathbb{C}(8)$ | $U(n)/O(n)$ | 0 |

# Some formulas

$$Cl_{p,q} \otimes Cl_{0,2} \cong Cl_{q,p+2} \qquad \{e_i\} \otimes \{\sigma_x, \sigma_z\} \cong \left\{ e_i \otimes \left( i\sigma_y \right), \sigma_x, \sigma_z \right\}$$

$$Cl_{0,2}: \ \{\sigma_x, \sigma_z\} \to \left\{ 1, \sigma_x, i\sigma_y, \sigma_z \right\} \to R(2): \text{ set of real } 2\times 2 \text{ matrices}$$

$$= Cl_{1,1}: \ \left\{ \sigma_x, i\sigma_y \right\}$$

$$Cl_{p,q} \otimes Cl_{2,0} \cong Cl_{q+2,p} \qquad \{e_i\} \otimes \left\{ i\sigma_y, i\tau_y\sigma_z \right\} \to \left\{ e_i \otimes i\tau_y\sigma_x, i\sigma_y, i\tau_y\sigma_z \right\}$$

$$Cl_{p,q} \otimes Cl_{1,1} \cong Cl_{p+1,q+1} \qquad \{e_i\} \otimes \left\{ \sigma_x, i\sigma_y \right\} \to \left\{ e_i \otimes \sigma_z, \sigma_x, i\sigma_y \right\}$$

$$Cl_{p,q} \otimes Cl_{0,4} \cong Cl_{p,q} \otimes Cl_{2,0} \otimes Cl_{0,2} \cong Cl_{q+2,p} \otimes Cl_{0,2} \cong Cl_{p,q+4}$$

$$\begin{aligned} Cl_{p,q+8} &\cong Cl_{p,q+4} \otimes Cl_{0,4} \cong Cl_{p,q+4} \otimes Cl_{2,0} \otimes Cl_{0,2} \cong Cl_{q+4,p+2} \otimes Cl_{2,0} \cong Cl_{p+4,q+4} \\ &\cong Cl_{p,q} \otimes Cl_{1,1} \otimes Cl_{1,1} \otimes Cl_{1,1} \otimes Cl_{1,1} \\ &\cong Cl_{p,q} \otimes R(16) \end{aligned}$$