Anderson Localization – Looking Forward

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Lecture1
September, 8, 2015

Outline

1. Introduction
2. Anderson Model; Anderson Metal and Anderson Insulator
3. Phonon-assisted hoping conductivity
4. Localization beyond the real space. Integrability and chaos.
5. Spectral Statistics and Localization
6. Many-Body Localization.
7. Many-Body Localization of the interacting fermions.
8. Many-Body localization of weakly interacting bosons.
9. Many-Body Localization and Ergodicity

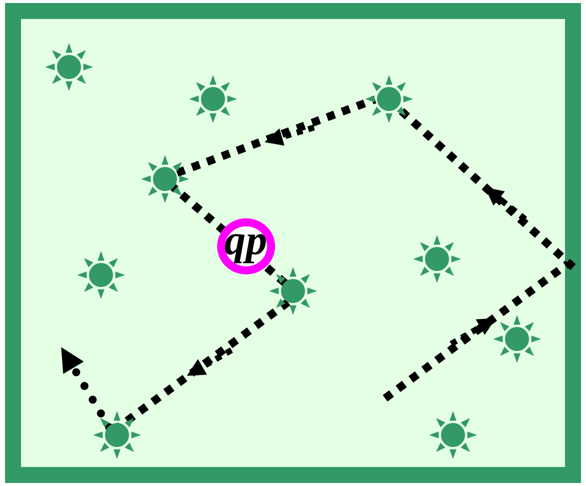
>50 years of Anderson Localization

PHYSICAL REVIEW VOLUME 109, NUMBER 5 MARCH 1, 1958

Absence of Diffusion in Certain Random Lattices

P. W. ANDERSON
Bell Telephone Laboratories, Murray Hill, New Jersey
(Received October 10, 1957)

This paper presents a simple model for such processes as spin diffusion or conduction in the "impurity band." These processes involve transport in a lattice which is in some sense random, and in them diffusion is expected to take place via quantum jumps between localized sites. In this simple model the essential randomness is introduced by requiring the energy to vary randomly from site to site. It is shown that at low enough densities no diffusion at all can take place, and the criteria for transport to occur are given.



…very few believed it [localization] at the time, and even fewer saw its importance; among those who failed to fully understand it at first was certainly its author…

Nobel Lecture

Nobel Lecture, December 8, 1977

Local Moments and Localized States

Part 1.

Introduction and History

Diffusion Equation

$$\frac{\partial \rho}{\partial t} - D\nabla^2 \rho = 0$$

$$\langle r^2 \rangle = Dt$$

Diffusion constant

$\rho(\vec{r}, t)$ Can be density of particles or energy density. It can also be the probability to find a particle at a given point at a given time

Einstein theory of Brownian motion, 1905

The diffusion equation is valid for any random walk provided that there is no memory (markovian process)

Diffusion Equation

$$\frac{\partial \rho}{\partial t} - D\nabla^2 \rho = 0$$

Einstein-Sutherland Relation for electric conductivity σ

William Sutherland
(1859-1911)

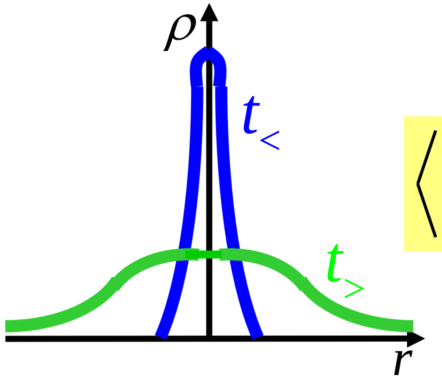
$$\sigma = e^2 D \nu \qquad \nu \equiv \frac{dn}{d\mu}$$

Density of states

If electrons would be degenerate and form a classical ideal gas

$$\nu = \frac{n_{tot}}{T}$$

Will a fluctuation (wave packet) spread ?



$$\left\langle r^2(t_>)\right\rangle \overset{?}{>} \left\langle r^2(t_<)\right\rangle$$

Einstein (1905): Random walk without memory

always diffusion

Diffusion Constant

$$\frac{\partial \rho}{\partial t} - D\nabla^2\rho = 0 \Rightarrow \left\langle r^2 \right\rangle = Dt$$

Anderson (1958): For quantum particles

not always

extended states $\left\langle r^2 \right\rangle \xrightarrow[t\to\infty]{} Dt$

localized states $\left\langle r^2 \right\rangle \xrightarrow[t\to\infty]{} const$

Basic Quantum Mechanics:

$$\left[-\frac{\nabla^2}{2m} + U(\boldsymbol{r}) - \epsilon_F\right]\psi_\alpha(\boldsymbol{r}) = \xi_\alpha \psi_\alpha(\boldsymbol{r})$$

Potential well

ξ_α -spectrum

Spectra

Continuous

Unbound states

$$\left|\psi_\alpha(\vec{r})\right|^2 \xrightarrow[L\to\infty]{} O\left(L^{-d}\right)$$

Extended states

Discrete

Bound states

$$\left|\psi_\alpha(\vec{r})\right|^2 \xrightarrow[|\vec{r}|\to\infty]{} O\left(e^{-|\vec{r}|/\varsigma}\right)$$

Localized states

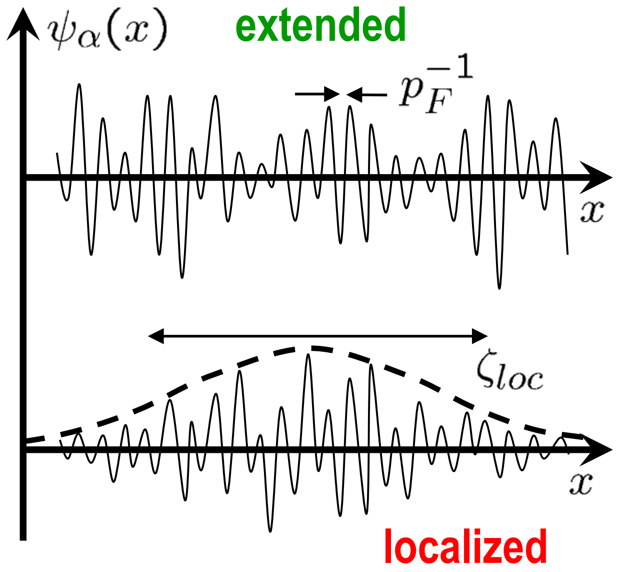
L System size

d Number of the spatial dimensions

Localization of single-particle wave-functions. Continuous limit:

Random potential

$$\left[-\frac{\boldsymbol{\nabla}^2}{2m} + U(\boldsymbol{r}) - \epsilon_F\right]\psi_\alpha(\boldsymbol{r}) = \xi_\alpha \psi_\alpha(\boldsymbol{r})$$



Experiment

Spin Diffusion

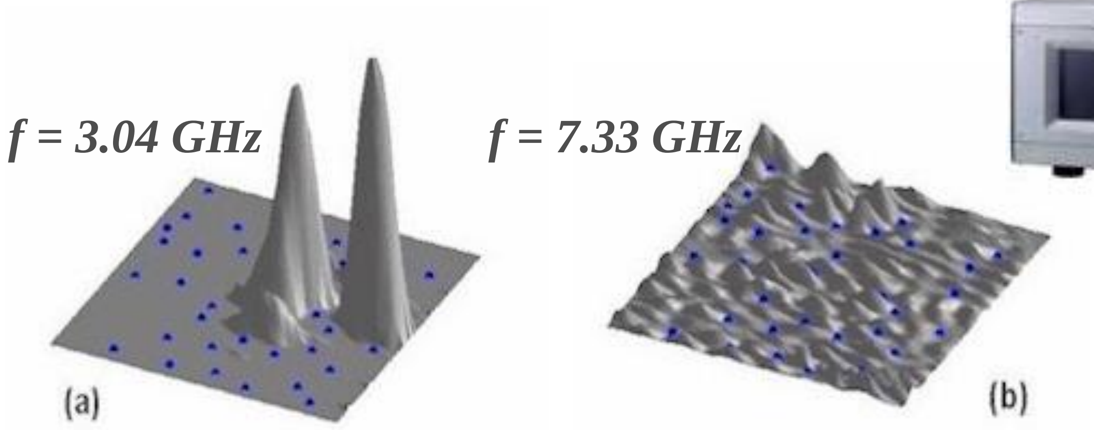
Feher, G., Phys. Rev. 114, 1219 (1959); Feher, G. & Gere, E. A., Phys. Rev. 114, 1245 (1959).

Microwave

Dalichaouch, R., Armstrong, J.P., Schultz, S.,Platzman, P.M. & McCall, S.L. "Microwave localization by 2-dimensional random scattering". *Nature* 354, 53-55, (1991).

Chabanov, A.A., Stoytchev, M. & Genack, A.Z. Statistical signatures of photon localization. *Nature* 404, 850-853 (2000).

Pradhan, P., Sridar, S, "Correlations due to localization in quantum eigenfunctions od disordered microwave cavities", PRL 85, (2000)



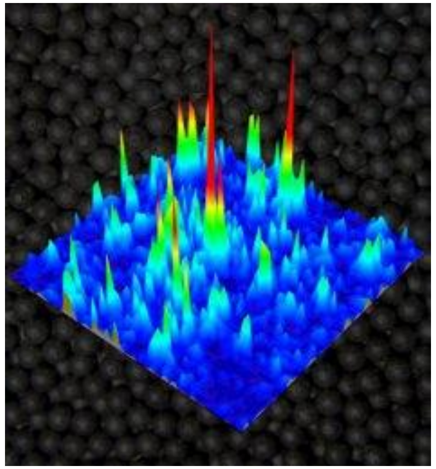
Localized *Extended*

Experiment

Localization of Ultrasound

Weaver, R.L. "Anderson localization of ultrasound".
Wave Motion 12, 129-142 (1990).

H. Hu, A. Strybulevych, J. H. Page, S. E. Skipetrov & B. A. van Tiggelen "**Localization of ultrasound in a three-dimensional elastic network**" Nature Phys. 4, 945 (**2008**).

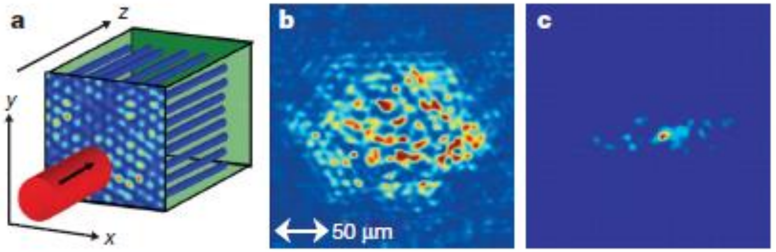


Localization of Light

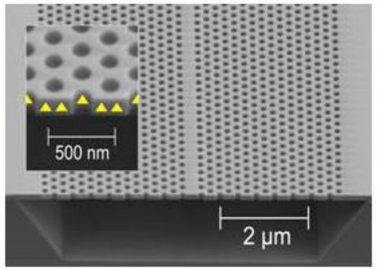
D. Wiersma, Bartolini, P., Lagendijk, A. & Righini R. "Localization of light in a disordered medium", Nature 390, 671-673 (1997).

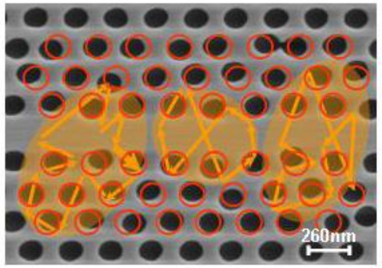
Scheffold, F., Lenke, R., Tweer, R. & Maret, G. "Localization or classical diffusion of light", Nature 398,206-270 (1999**).**

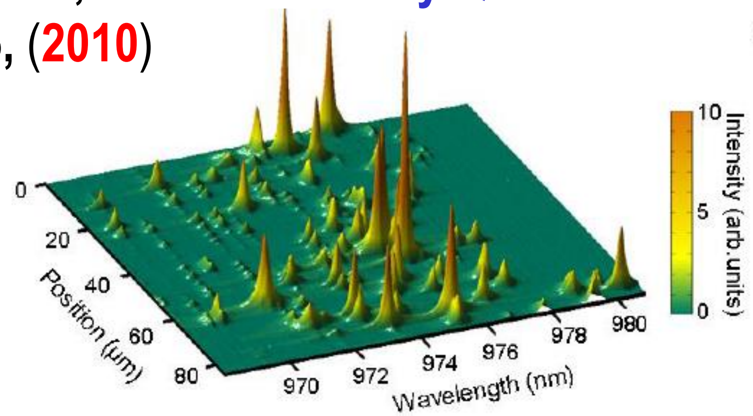
Schwartz, T., Bartal, G., Fishman, S. & Segev, M. "Transport and Anderson localization in disordered two dimensional photonic lattices". Nature 446, 52-55 (**2007**).



L.Sapienza, H.Thyrrestrup, S.Stobbe, P. D.Garcia, S.Smolka, P.Lodahl "Cavity Quantum Electrodynamics with Anderson localized Modes" Science 327, 1352-1355, (**2010**)

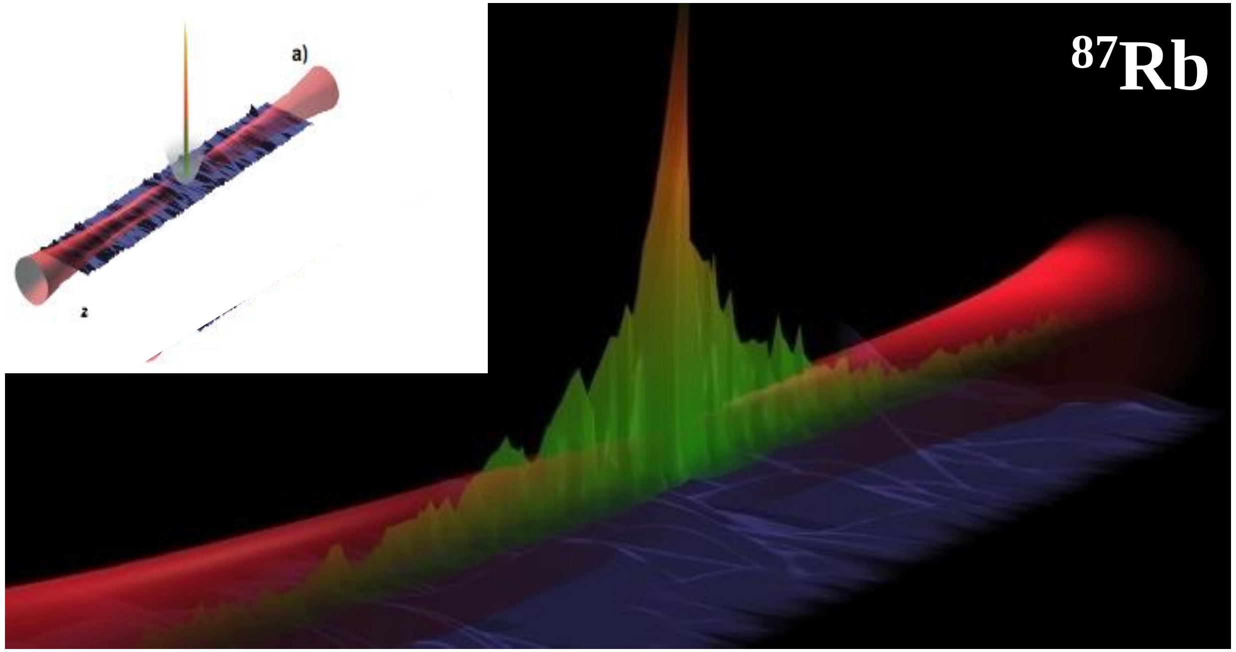






Localization of cold atoms

Billy et al. “Direct observation of Anderson localization of matter waves in a controlled disorder”. Nature 453, 891- 894 (2008).



Roati et al. “Anderson localization of a non-interacting Bose-Einstein condensate“. Nature 453, 895-898 (2008).

What about charge transport ?

Problem: electrons interact with each other

Einstein Relation (1905)

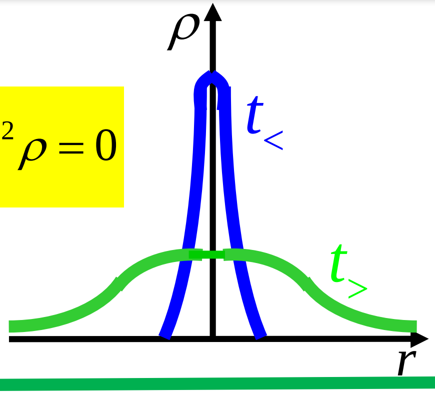
$$\sigma = e^2 D \frac{dn}{d\mu}$$

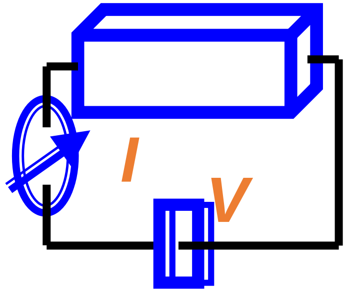
Conductivity

Diffusion Constant

Density of states

$$\frac{\partial \rho}{\partial t} - D\nabla^2 \rho = 0$$





$$G = \left(\frac{I}{V}\right)_{V=0}$$ Conductance

$$s = G\frac{L}{A}$$ Conductivity

At zero temperature

Extended states
- metal

$$G \propto L^{d-2};$$

$$s \xrightarrow[L\to\infty]{} const; \quad D \xrightarrow[L\to\infty]{} const$$

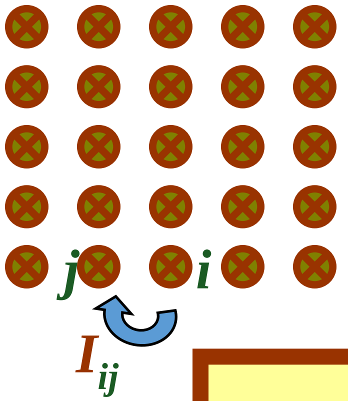
Localized states
- insulator:

$$G \mu e^{-L/x}; \quad s = 0; \quad D = 0$$

Part 2.

Anderson Model;
Anderson Metal
and
Anderson Insulator

Anderson Model



- *Lattice - tight binding model*
- *Onsite energies* ε_i - *random*
- *Hopping matrix elements* I_{ij}

$-W < \varepsilon_i < W$

uniformly distributed

$$I_{ij} = \begin{cases} I & i \text{ and } j \text{ are nearest neighbors} \\ 0 & \text{otherwise} \end{cases}$$

Anderson Transition

$I < I_c$

Insulator

All eigenstates are localized

Localization length ξ

$I > I_c$

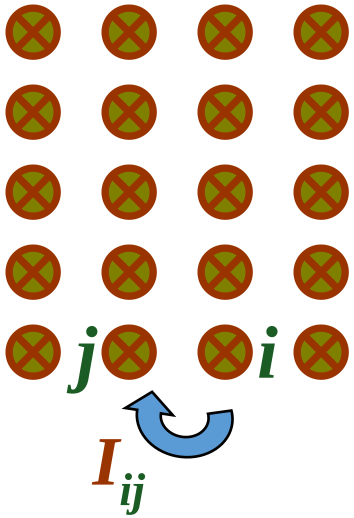
Metal

There appear states extended all over the whole system

One-dimensional Anderson Model

$$\hat{H} = \begin{pmatrix} \varepsilon_1 & I & \ddots & 0 \\ I & \ddots & \ddots & I \\ \ddots & \ddots & & \\ 0 & & I & \varepsilon_N \end{pmatrix}$$

Q: Why arbitrary weak hopping I is not sufficient for the existence of the diffusion ?



Einstein (1905): Marcovian (no memory) process → diffusion

Quantum mechanics is not marcovian
There is memory in quantum propagation !

Why ?

A: Quantum Interference

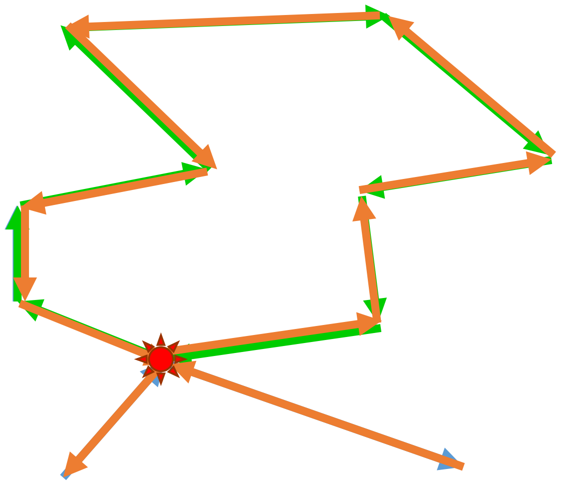
Quantum mechanics is not marcovian

There is memory in quantum propagation!

Why?

A: Quantum Interference

Memory!



Quantum mechanics is not marcovian
There is memory in quantum propagation!

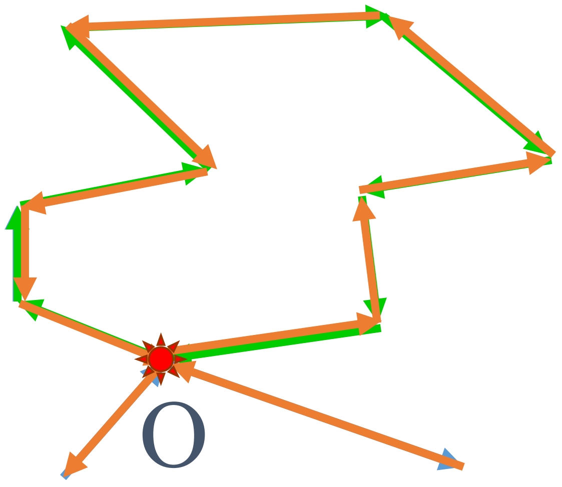
Why?

A: Quantum Interference

Memory!

$$\phi = \oint \vec{p} d\vec{r}$$

Phase accumulated when traveling along the loop



The particle can go around the loop in two directions

$$\varphi_1 = \varphi_2$$

Constructive interference ⟹ probability of the return to the origin gets enhanced ⟹ quantum corrections reduce the diffusion constant.

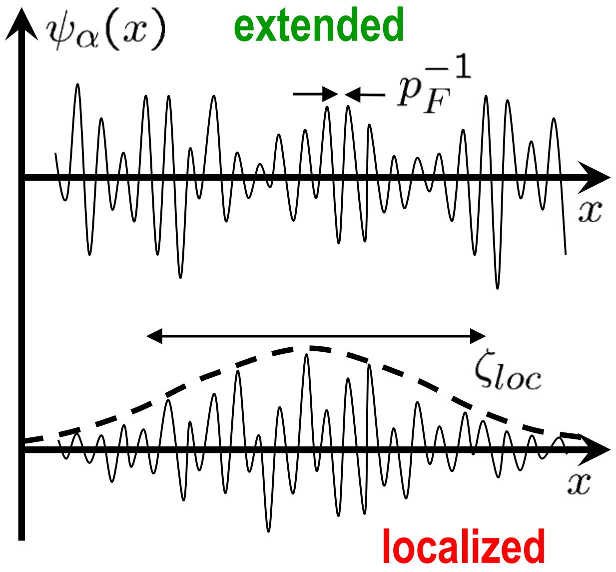
Tendency towards localization

WEAK LOCALIZATION

Localization of single-particle wave-functions. Continuous limit:

Random potential

$$\left[-\frac{\nabla^2}{2m} + U(\boldsymbol{r}) - \epsilon_F\right]\psi_\alpha(\boldsymbol{r}) = \xi_\alpha \psi_\alpha(\boldsymbol{r})$$

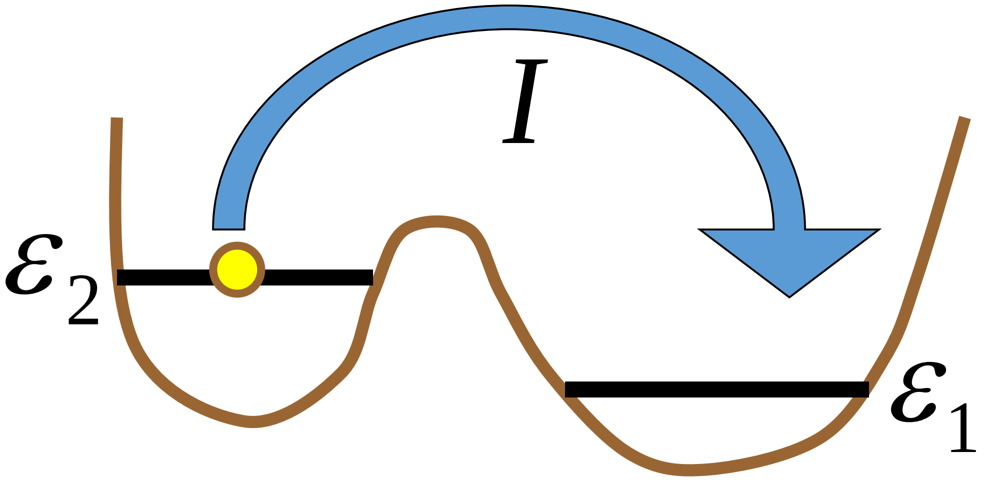


d=1; **All states are localized**

d=2; **All states are localized**

d >2; **Anderson transition**

Two-site Anderson model = Two-well potential



Hamiltonian

$$\hat{H} = \begin{pmatrix} \varepsilon_1 & I \\ I & \varepsilon_2 \end{pmatrix} \xrightarrow{\text{diagonalize}} \hat{H} = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}$$

$$E_2 - E_1 = \sqrt{(\varepsilon_2 - \varepsilon_1)^2 + I^2}$$

$$\hat{H} = \begin{pmatrix} \varepsilon_1 & I \\ I & \varepsilon_2 \end{pmatrix} \quad \xrightarrow{\text{diagonalize}} \quad \hat{H} = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}$$

$$E_2 - E_1 = \sqrt{(\varepsilon_2 - \varepsilon_1)^2 + I^2} \approx \begin{matrix} \varepsilon_2 - \varepsilon_1 & \quad \varepsilon_2 - \varepsilon_1 >> I \\ I & \quad \varepsilon_2 - \varepsilon_1 << I \end{matrix}$$

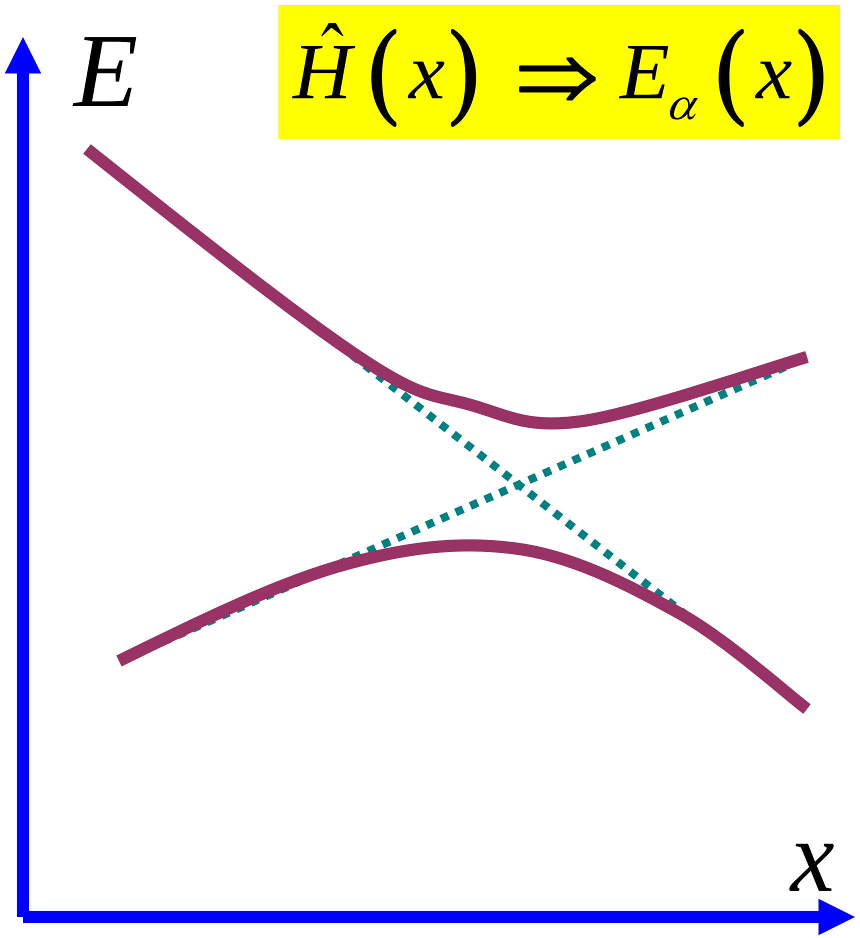
von Neumann & Wigner “noncrossing rule”

Level repulsion

v. Neumann J. & Wigner E. 1929 Phys. Zeit. v.30, p.467

Arnold V.I., *Mathematical Methods of Classical Mechanics (Springer-Verlag: New York), Appendix 10, 1989*

In general, a multiple spectrum in typical families of quadratic forms is observed only for two or more parameters, while in one-parameter families of general form the spectrum is simple for all values of the parameter. Under a change of parameter in the typical one-parameter family the eigenvalues can approach closely, but when they are sufficiently close, it is as if they begin to repel one another. The eigenvalues again diverge, disappointing the person who hoped, by changing the parameter to achieve a multiple spectrum.



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What about the eigenfunctions ?

$$\hat{H} = \begin{pmatrix} \varepsilon_1 & I \\ I & \varepsilon_2 \end{pmatrix}$$

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What about the eigenfunctions ?

$$\phi_1, \varepsilon_1; \phi_2, \varepsilon_2 \quad \Leftarrow \quad \psi_1, E_1; \psi_2, E_2$$

$$|\varepsilon_2 - \varepsilon_1| >> I$$

$$\psi_{1,2} = \phi_{1,2} + O\left(\frac{I}{|\varepsilon_2 - \varepsilon_1|}\right)\phi_{2,1}$$

Off-resonance

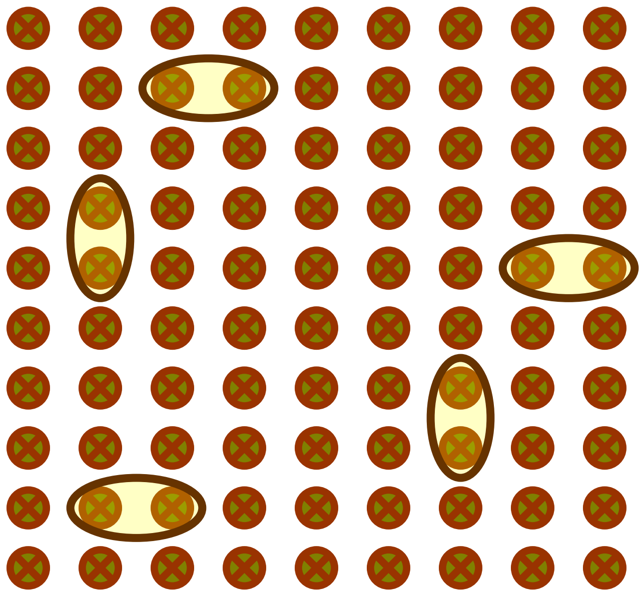
Eigenfunctions are close to the original on-site wave functions

$$|\varepsilon_2 - \varepsilon_1| << I$$

$$\psi_{1,2} \approx \phi_{1,2} \pm \phi_{2,1}$$

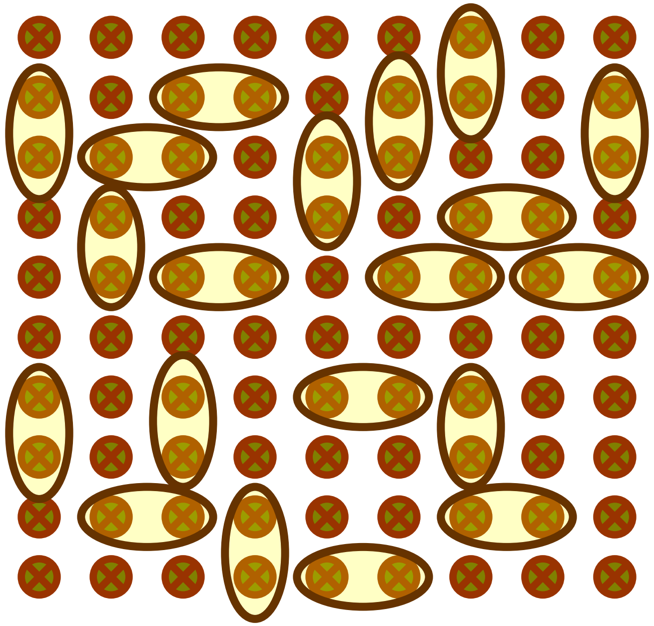
Resonance

In both bonding and anti-bonding eigenstates the probability is equally shared between the sites



Anderson insulator

Few isolated resonances

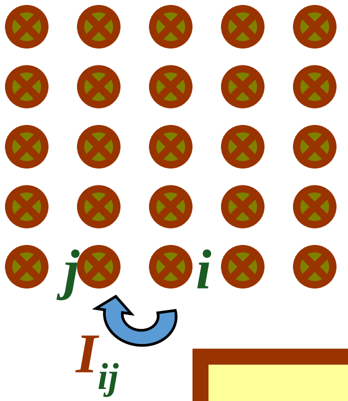


Anderson metal

Many resonances and they overlap

Transition: Typically each site is in the resonance with some other one

Anderson Model



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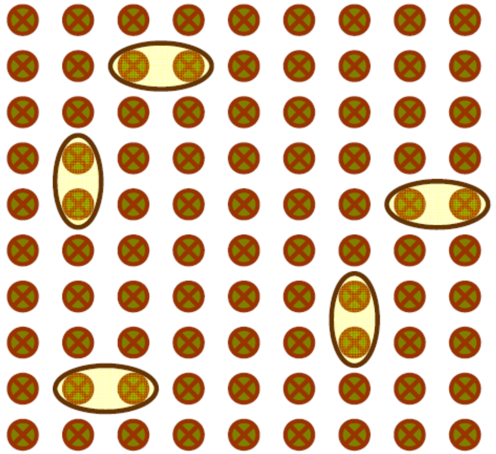
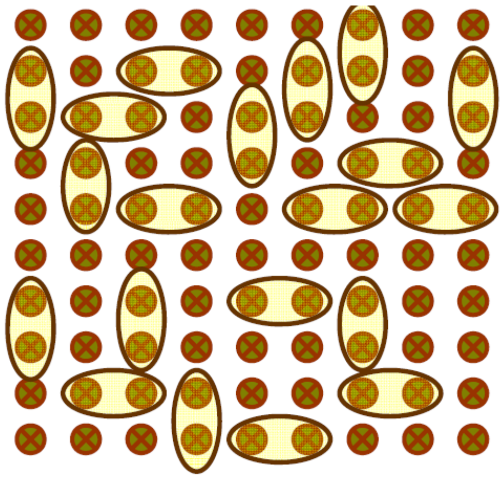


Anderson insulator
Few isolated resonances



Anderson metal
There are many resonances and they overlap

Condition for Localization:

$$I < \frac{\text{energy mismatch}}{\text{\# of n.neighbors}}$$

Transition: Typically each site is in the resonance with some other one

energy mismatch $= \left|\varepsilon_i - \varepsilon_j\right|_{typ} = W$

# of nearest neighbors $= 2d$

A bit more precise:

$$\frac{I_c}{W} \simeq \left(\frac{1}{2d}\right)\left(\frac{1}{\ln d}\right)$$

Logarithm is due to the resonances, which are not nearest neighbors

Condition for Localization:

$$\frac{I_c}{W} \simeq \left(\frac{1}{2d}\right)\left(\frac{1}{\ln d}\right)$$

Q: Is it correct?

A1: For low dimensions - NO. $I_c = \infty$ for $d = 1, 2$
All states are localized. Reason - loop trajectories

A2: Works better for larger dimensions $d > 2$

A3: Is exact on the Bethe lattice

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Rule of the thumb

At the localization transition a typical site is in resonance with another one

At the localization transition the hoping matrix element is of the order of the typical energy mismatch divided by the number of nearest neighbors

Anderson's recipe:

Consider an open system (Anderson Model).
Particle escape → "broadening" of each eigenstate

$$E \to E + i\Gamma \,; \qquad \Gamma \equiv \mathrm{Im}\,\Sigma$$

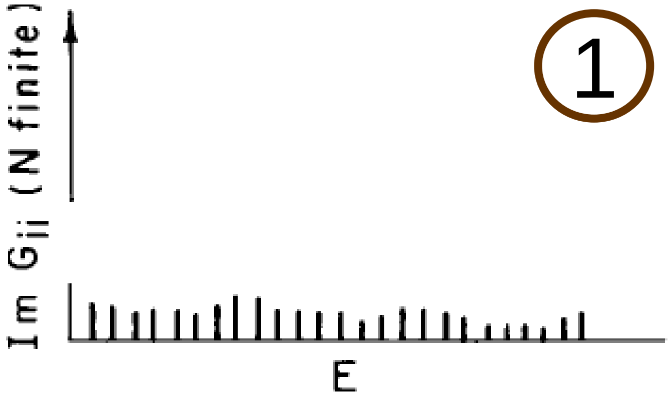
Γ - scape rate (inverse dwell time)

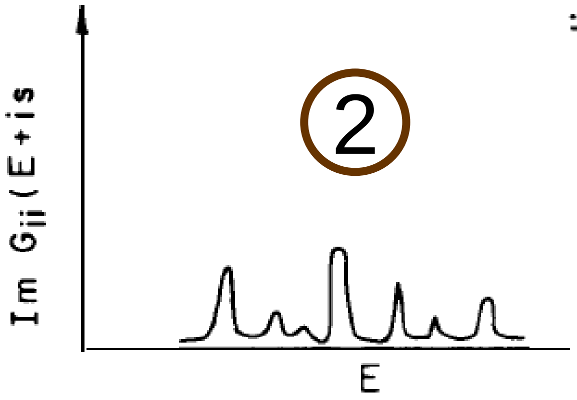
$$I = 0 \quad \Rightarrow \quad \Gamma = 0$$

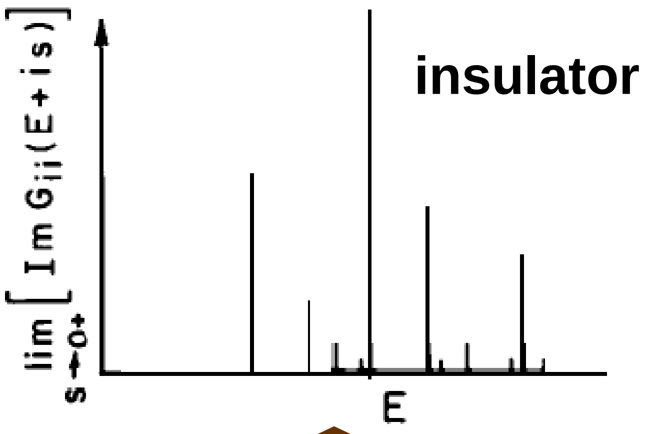
States localized inside the system – small Γ
Extended states – large Γ

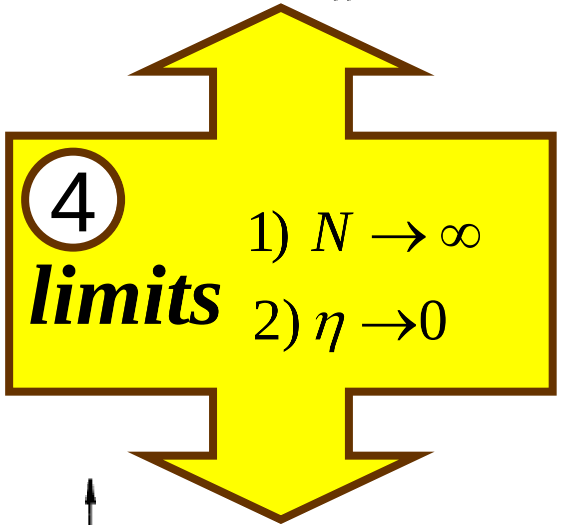
Anderson's recipe:

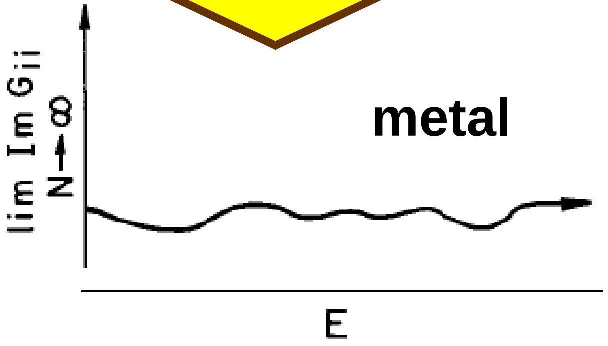
1. take descrete spectrum E_μ of H_0
2. Add an infinitesimal Im part $i\eta$ to E_μ
3. Evaluate $Im\Sigma_\mu$





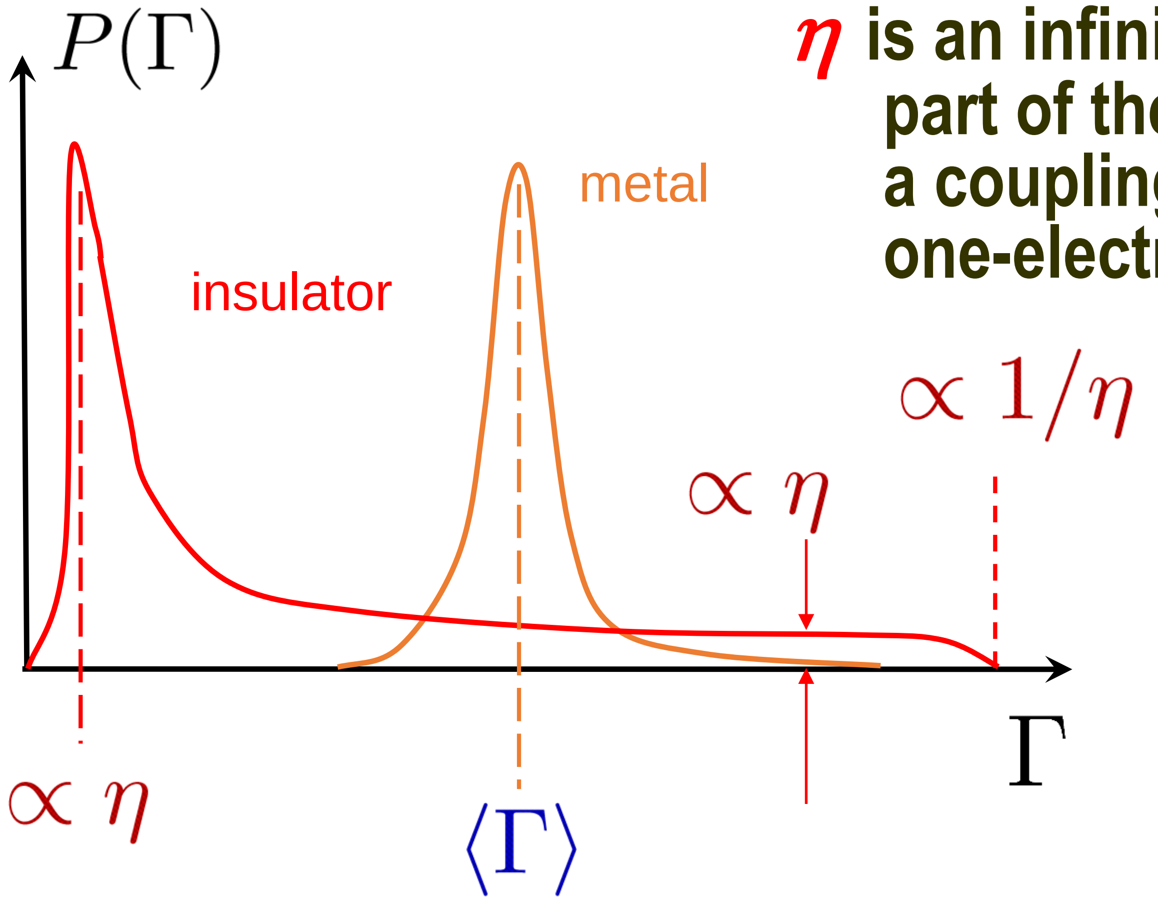






4. take limit $\eta \to 0$ but only after $N \to \infty$
5. "What we really need to know is the *probability distribution* of $Im\Sigma$, *not* its average…" !

Probability Distribution of *Γ=Im Σ*



η **is an infinitesimal width (*Im* part of the self-energy due to a coupling with a bath) of one-electron eigenstates**

Look for:

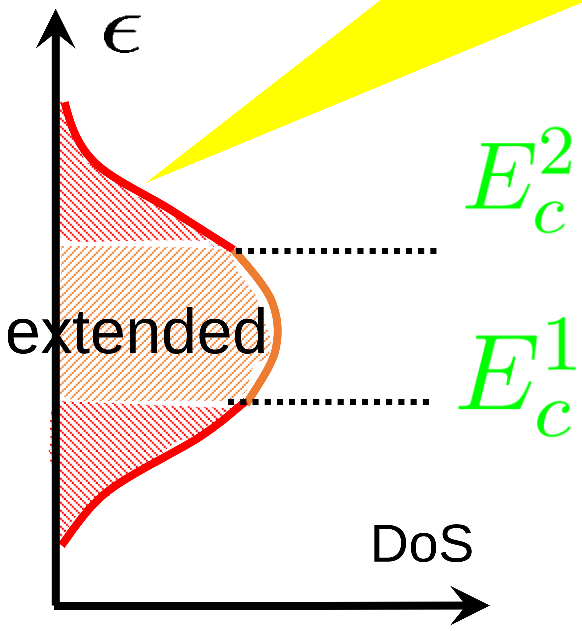
$$\lim_{\eta \to +0} \lim_{V \to \infty} P(\Gamma > 0) = \begin{cases} > 0; & metal \\ 0; & insulator \end{cases}$$

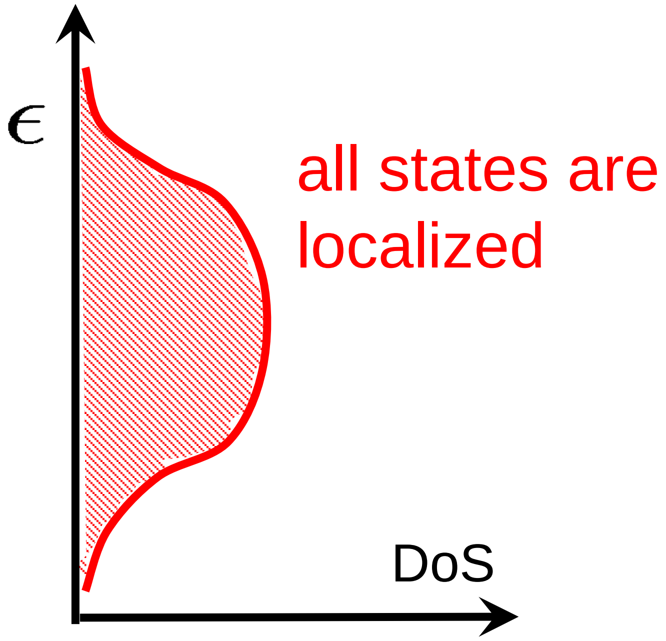
Anderson Transition

$I > I_c$

$I < I_c$

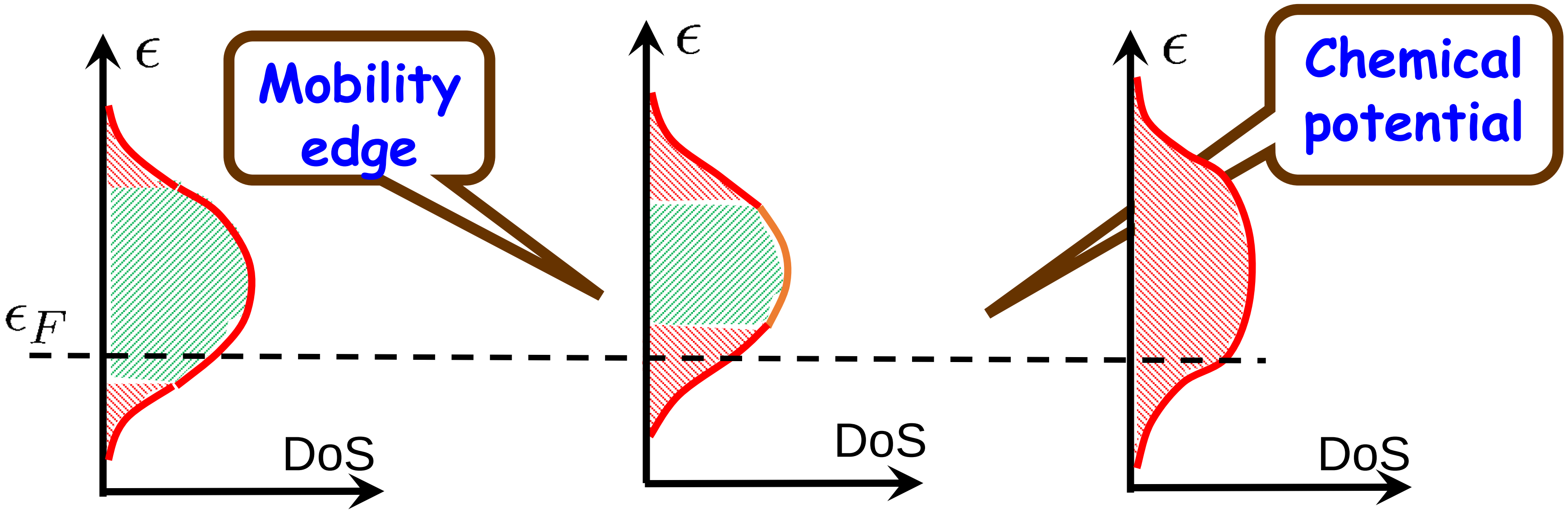
localized and extended never coexist!





E_c - mobility edges

Temperature dependence of the conductivity
one-electron picture



$\sigma(T \to 0) > 0$ $\quad$ $\sigma(T) \propto e^{-\frac{E_c - \varepsilon_F}{T}}$ $\quad$ $\sigma(T) = 0 \;\; \forall T$

there are extended states — $I > I_c$

all states are localized — $I < I_c$

Part 3.

Localization beyond real space

Integrability and chaos

VOLUME 49 23 AUGUST 1982 NUMBER 8

Chaos, Quantum Recurrences, and Anderson Localization

Shmuel Fishman, D. R. Grempel, and R. E. Prange
Department of Physics and Center for Theoretical Physics, University of Maryland, College Park, Maryland 20742
(Received 6 April 1982)

A periodically kicked quantum rotator is related to the Anderson problem of conduction in a one-dimensional disordered lattice. Classically the second model is always chaotic, while the first is chaotic for some values of the parameters. With use of the Anderson-model result that all states are localized, it is concluded that the *local* quasienergy spectrum of the rotator problem is discrete and that its wave function is almost periodic in time. This allows one to understand on physical grounds some numerical results recently obtained in the context of the rotator problem.

Localization in the angular momentum space

Quantum and Classical Dynamical Systems

Large number $d \gg 1$ of the degrees of freedom

Conventional Boltzmann-Gibbs Statistical Physics

Equipartition Postulate

Ergodicity: time average = space (ensemble) average

Chaos

Hamiltonian $H(\{p_i, q_i\})$

?

$$H = H_0 + \lambda V$$

Integrable Systems

d degrees of freedom
d integrals of motion

Ergodicity is violated

Invariant tori dimension d

Hamiltonian $H_0(\{p_i, q_i\})$

Energy shell, dimension $2d-1$

Classical Dynamics

λ ← Ergodicity / Equipartition ——— KAM region / Arnold diffusion / Non-ergodic → 0

Fermi, Pasta, Ulam system
(connected nonlinear oscillators)

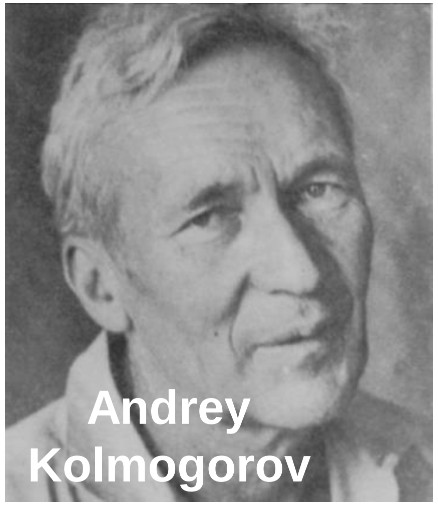
Solar system

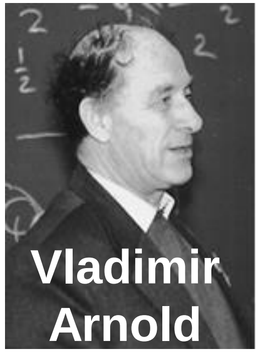
. . .

Quantum Dynamics ???

Kolmogorov – Arnold – Moser (KAM) theory

A.N. Kolmogorov,
Dokl. Akad. Nauk SSSR, 1954.
Proc. 1954 Int. Congress of Mathematics, North-Holland, 1957





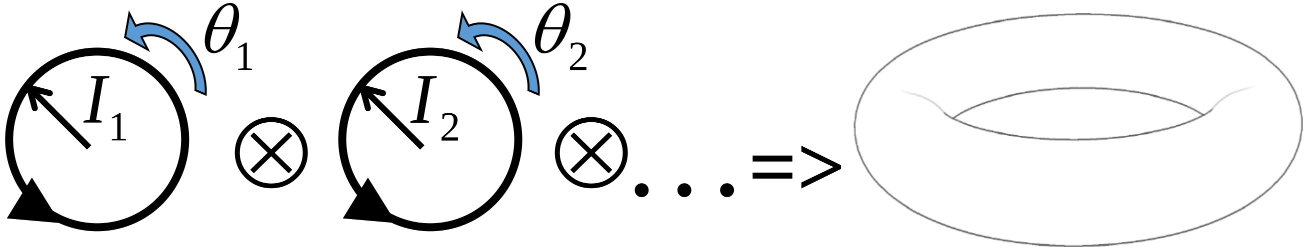
Integrable classical Hamiltonian

$\hat{H}_0$ $d > 1$:

Separation of variables: d sets of action-angle variables

$$I_1, \theta_1 = 2\pi\omega_1 t; .., I_2, \theta_2 = 2\pi\omega_2 t; ..$$

Quasiperiodic motion: set of the frequencies, $\omega_1, \omega_2, .., \omega_d$, which are in general incommensurate. Actions I_i are integrals of motion $\partial I_i / \partial t = 0$



tori

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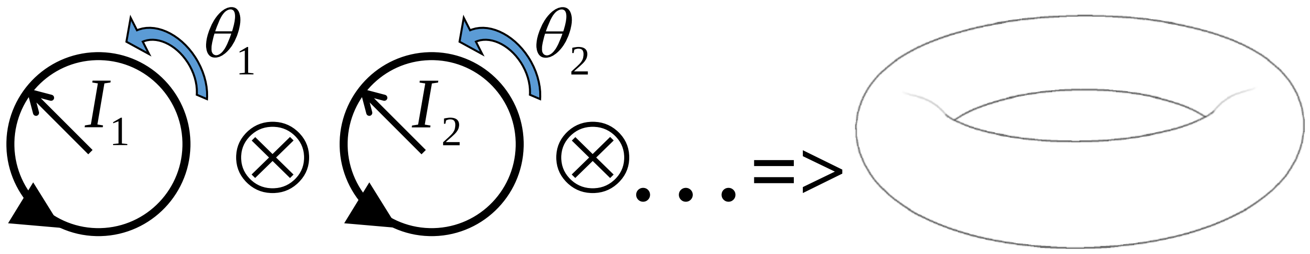


Given the set of the integrals of motion $\{I_\mu\}$ all trajectories belong to a torus

Q: Will an arbitrary weak perturbation $\hat{V}$ of an integrable Hamiltonian $\hat{H}_0$ destroy the tori and make the motion ergodic (when each point at the energy shell will be reached sooner or later) **?**

A: **Most of the tori survive weak and smooth enough perturbations**

KAM theorem

Classical Dynamics: from KAM to Chaos

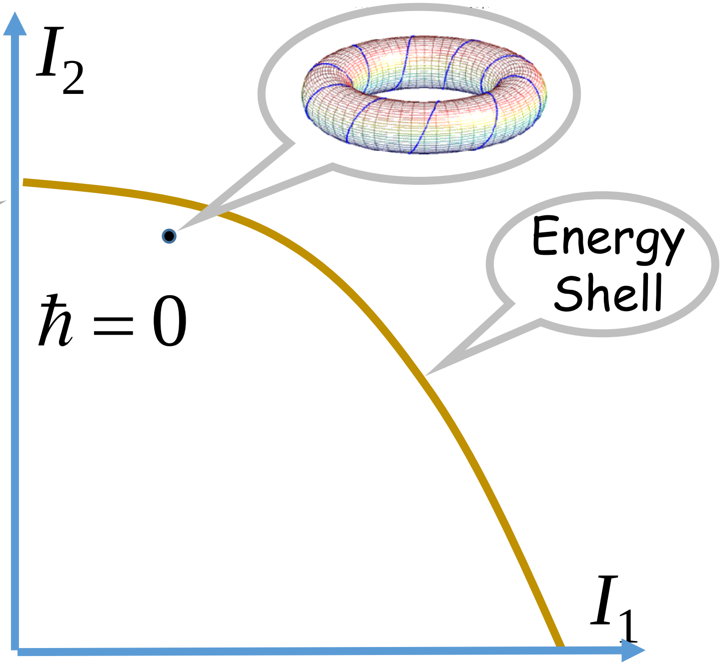
$$H = H_0 + \lambda V$$

Quantum Dynamics: Many – Body Localization

Classical, **d>>1** degrees of freedom

$I_1, I_2, \ldots, I_d$
integrals of motion

Space of the integrals of motion

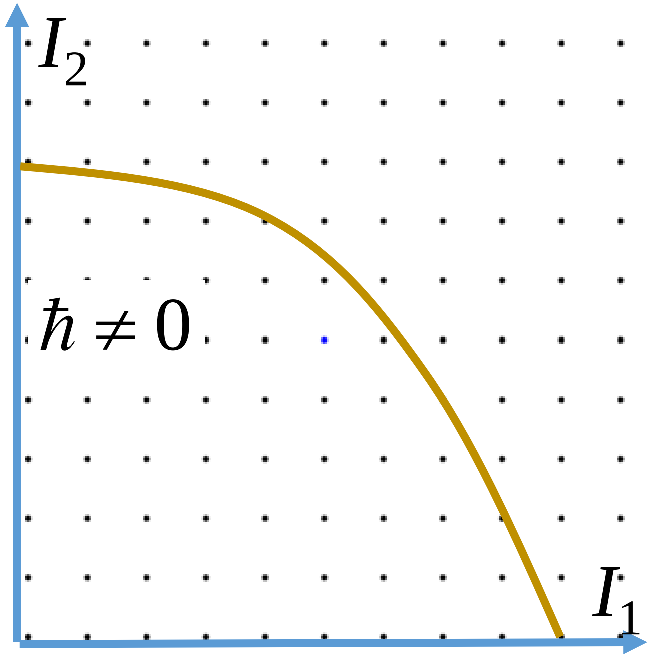


Quantum, **d>>1** degrees of freedom

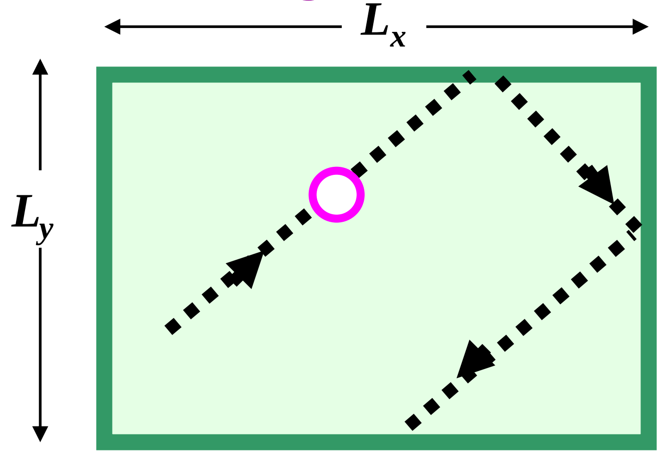
Integrals of motion are quantized – quantum numbers

Form a “lattice”.

Sites of the “lattice” – eigenstate of the integrable system

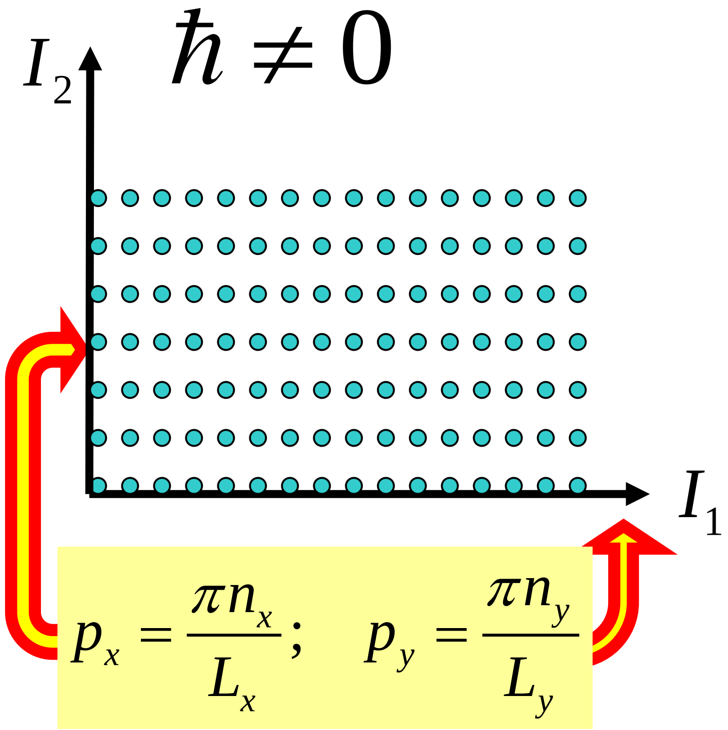


Rectangular billiard



Two integrals of motion

$$I_1 = p_x; \quad I_1 = p_y$$



Energy shell: $p_x^2 + p_y^2 = 2mE$

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$$H = H_0 + \lambda V$$

Quantum Dynamics: Many – Body Localization

Classical, $d>>1$ degrees of freedom

$I_1, I_2, \ldots, I_d$
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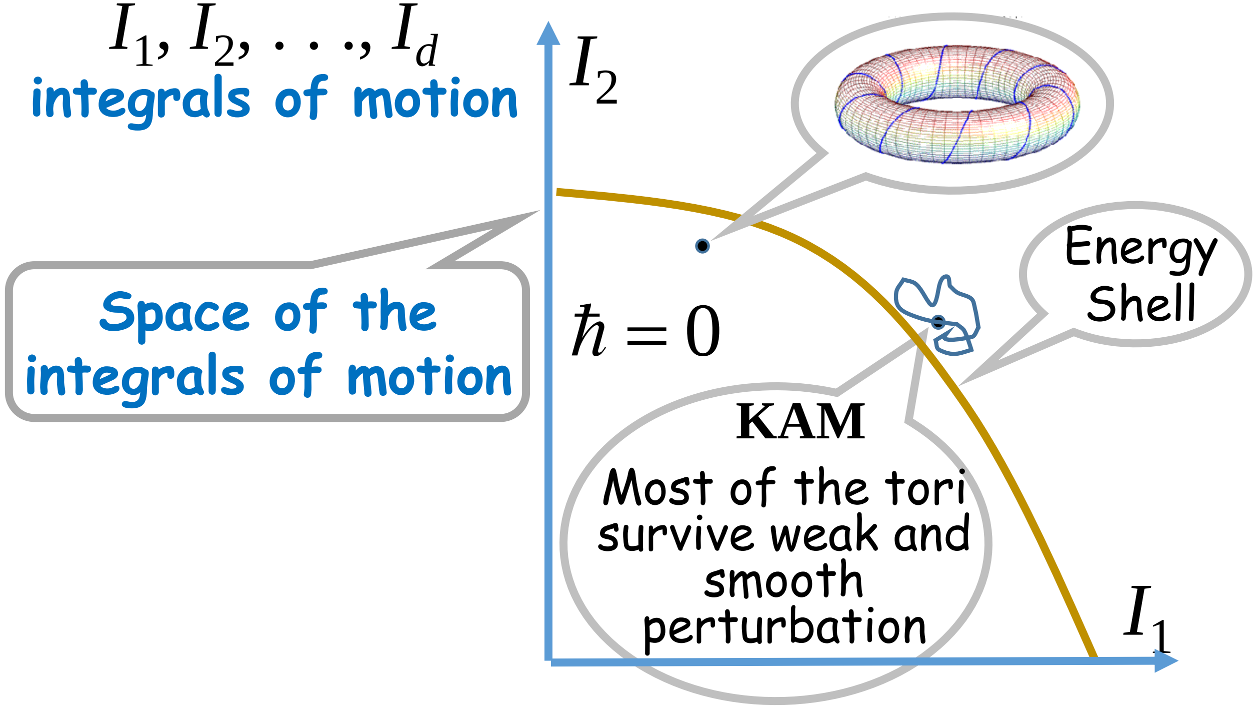

Quantum, $d>>1$ degrees of freedom

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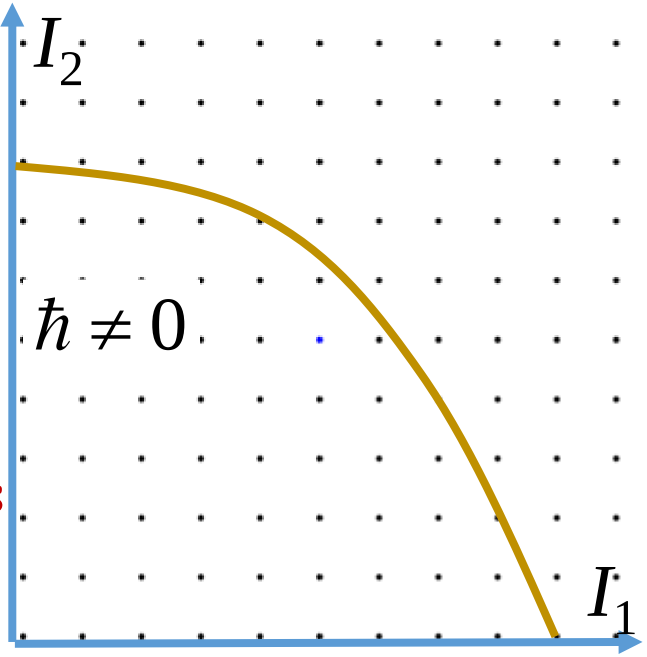
Sites of the "lattice" – eigenstate of the integrable system

Perturbation – coupling of the different sites of the "lattice" - bonds



Classical Dynamics: from KAM to Chaos

$$H = H_0 + \lambda V$$

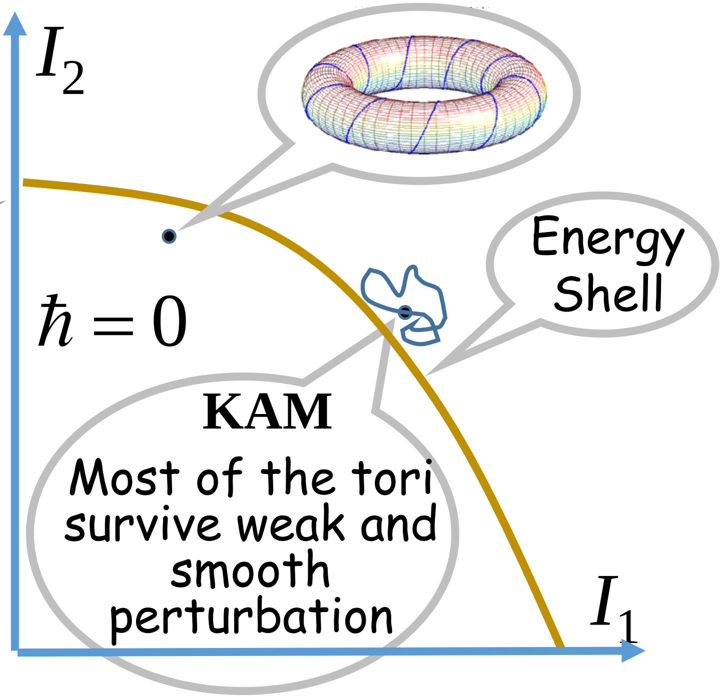
Quantum Dynamics: Many – Body Localization

Classical, $d>>1$ degrees of freedom

$I_1, I_2, \ldots, I_d$

integrals of motion

Space of the integrals of motion



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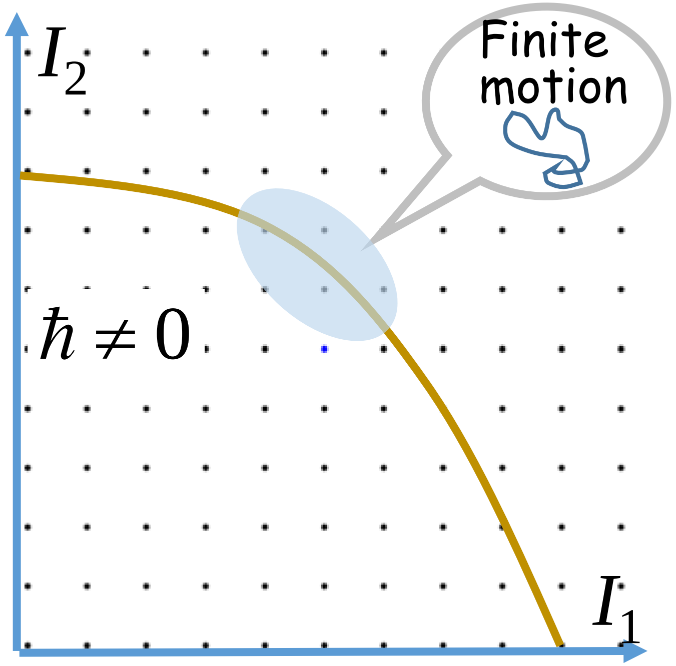
Wave functions localized in the space of quantum numbers



Classical Dynamics: from KAM to Chaos

$$H = H_0 + \lambda V$$

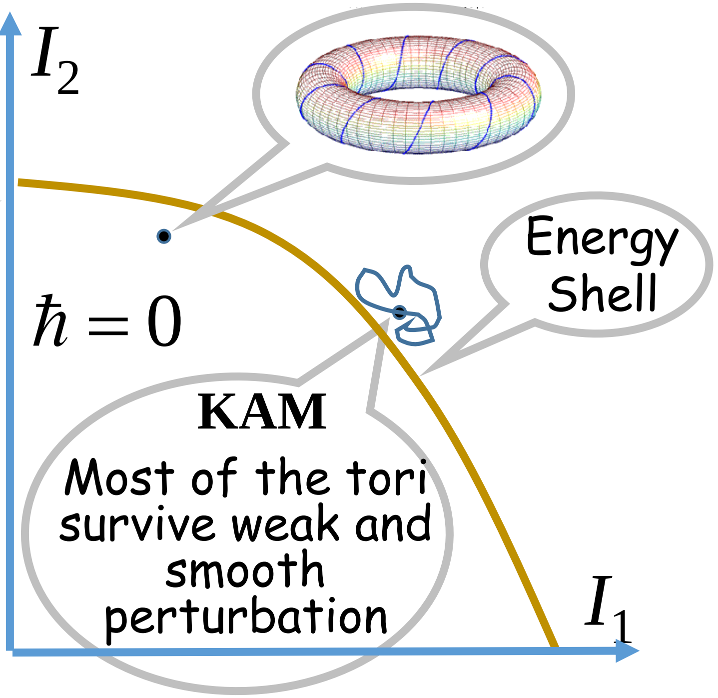
Quantum Dynamics: Many - Body Localization

Classical, $d>>1$ degrees of freedom

$I_1, I_2, \ldots, I_d$

integrals of motion

Space of the integrals of motion



Quantum, $d>>1$ degrees of freedom

Integrals of motion are quantized - quantum numbers

Form a "lattice".

Sites of the "lattice" - eigenstate of the integrable system

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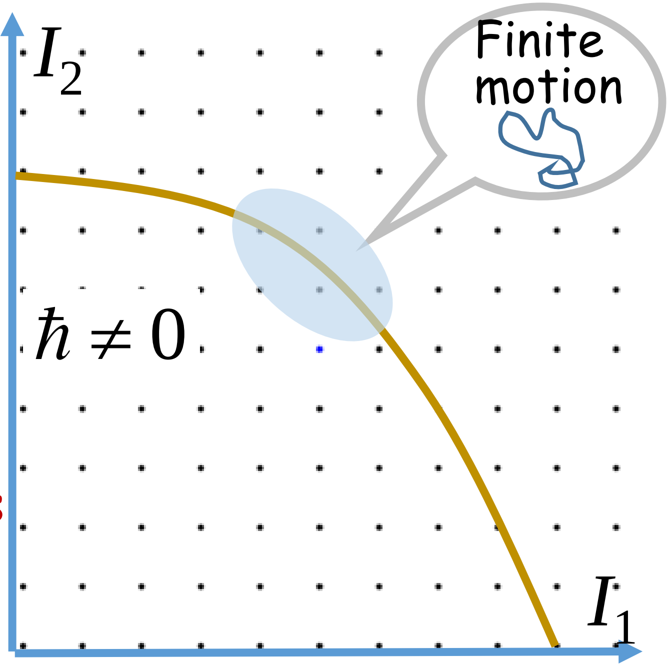
Quantum random walk

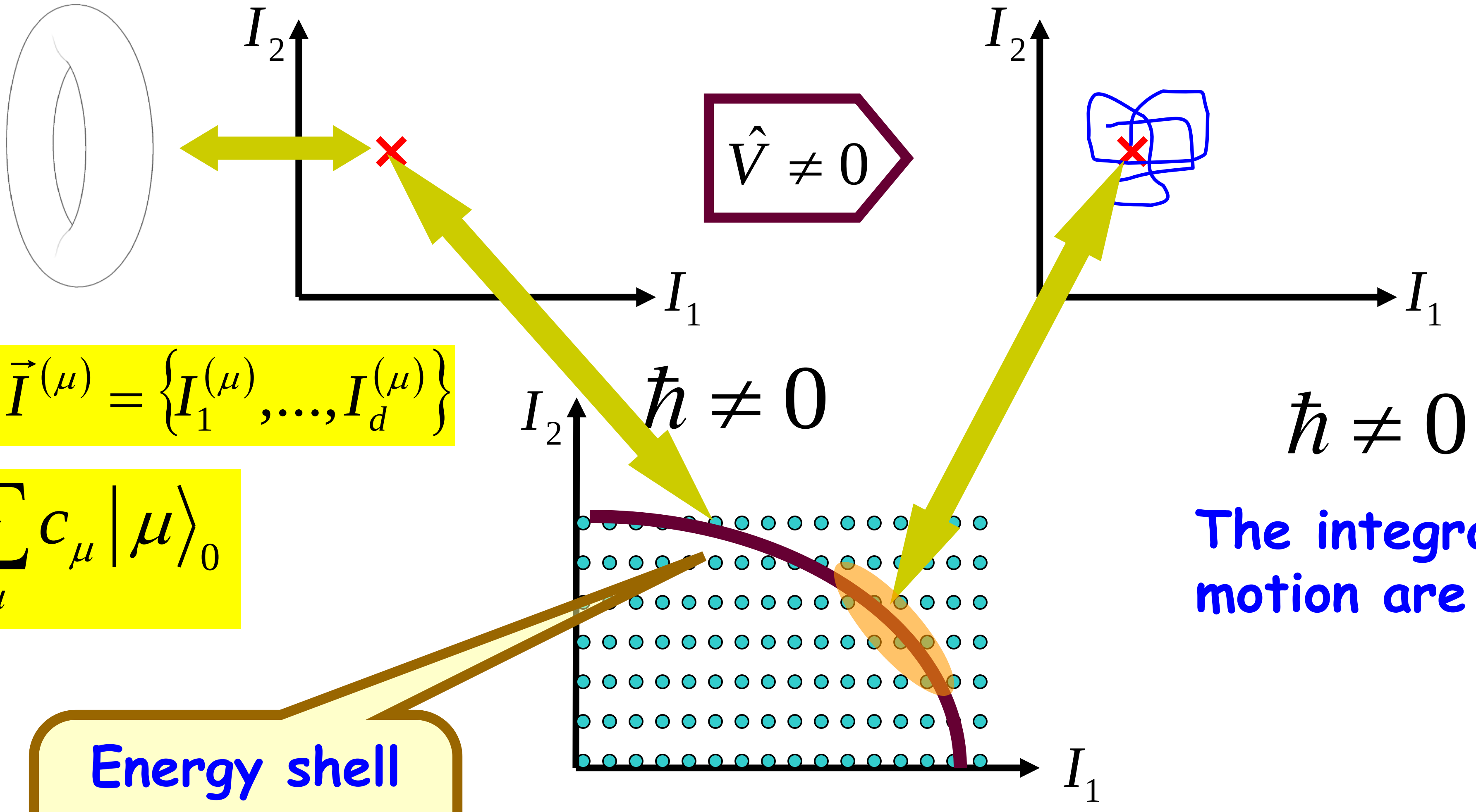
⇩

Localization?



Many - Body Localization is an analog of the Anderson Localization in a finite-dimensional space of a quantum particle subject to a random potential

KAM theorem: Most of the tori survive weak and smooth enough perturbations



$$|\mu\rangle_0 = \left|\vec{I}^{(\mu)}\right\rangle$$

$$\vec{I}^{(\mu)} = \left\{I_1^{(\mu)}, \ldots, I_d^{(\mu)}\right\}$$

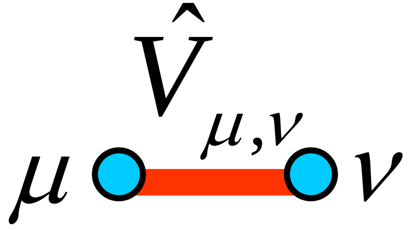
$$|\psi\rangle = \sum_\mu c_\mu |\mu\rangle_0$$

Energy shell

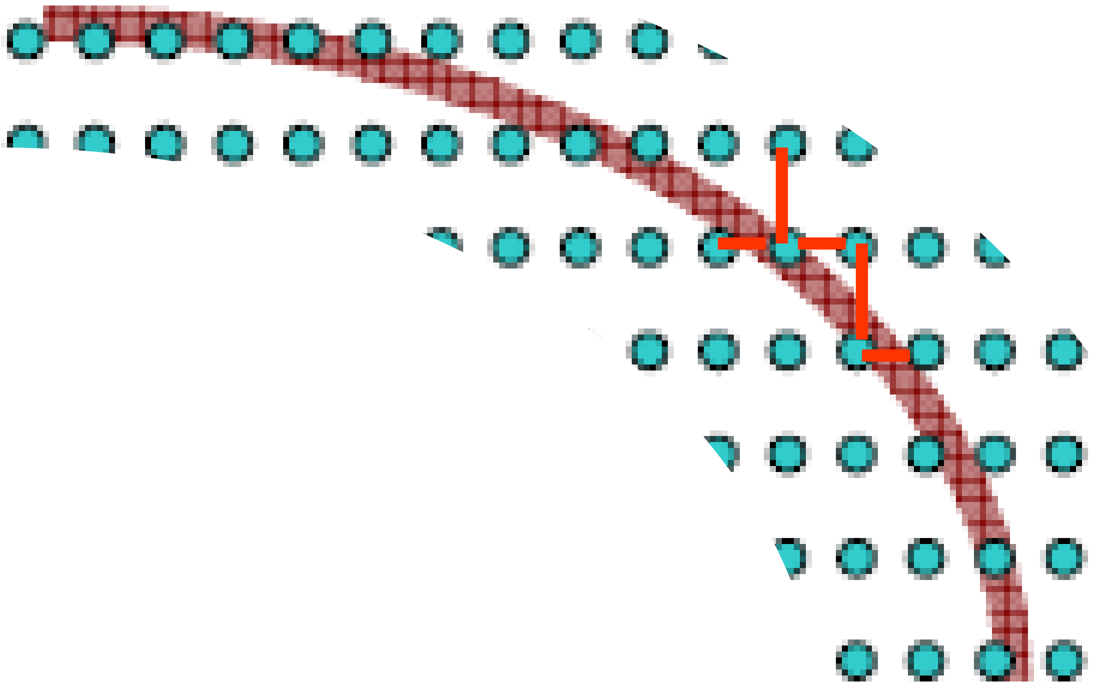
$$\frac{p_x^2 + p_y^2}{2m} = E$$

$$\hbar \neq 0$$

The integrals of motion are quantized



Matrix element of the perturbation



$$|\mu\rangle_0 = \left|\vec{I}^{(\mu)}\right\rangle$$

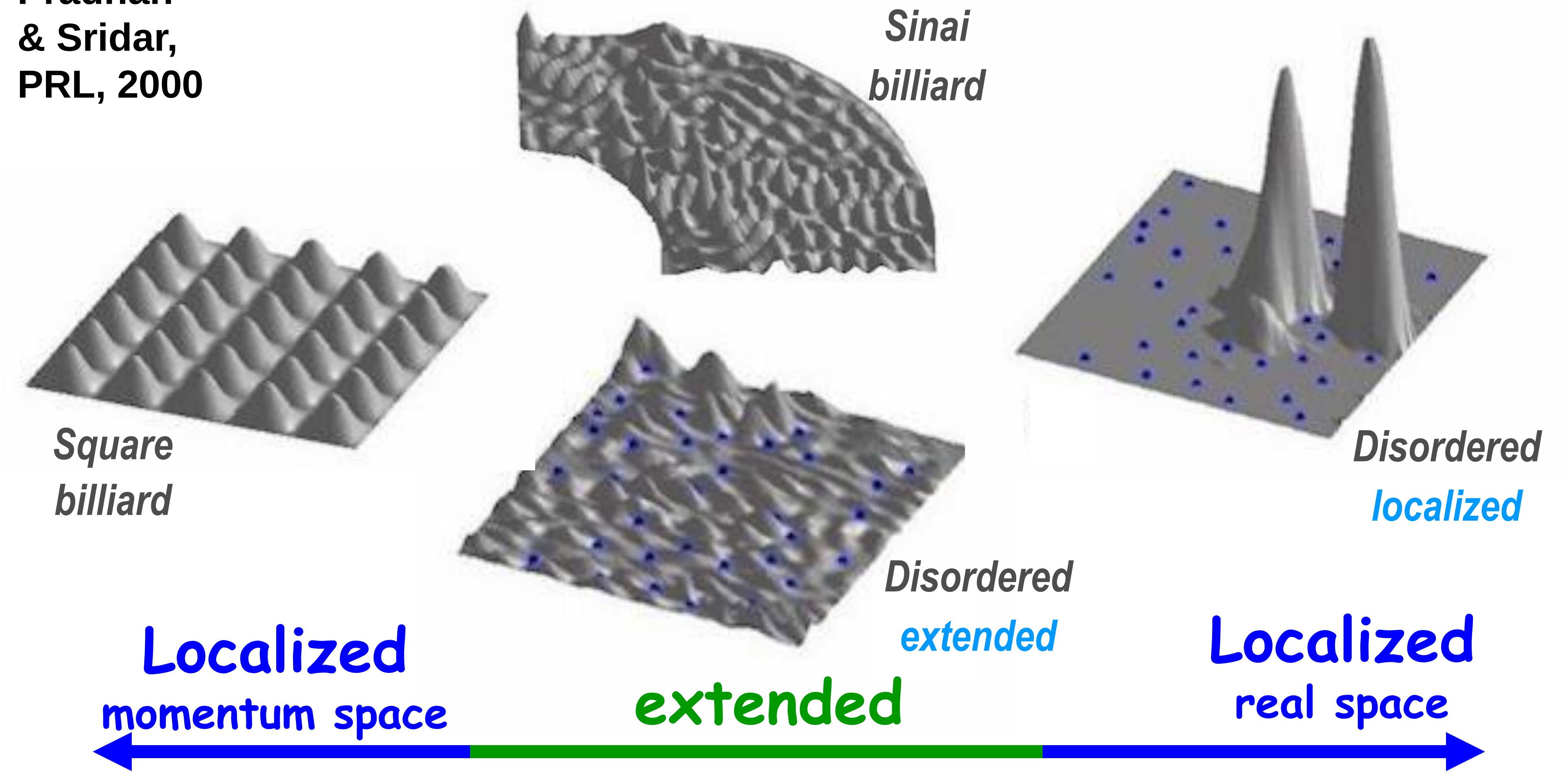
$$\vec{I}^{(\mu)} = \left\{I_1^{(\mu)}, \ldots, I_d^{(\mu)}\right\}$$

Anderson Model !

AL hops are local – one can distinguish "near" and "far"

KAM perturbation is smooth enough

Pradhan
& Sridar,
PRL, 2000



Glossary

Classical	Quantum
Integrable $H_0 = H_0(\vec{I}); \quad \partial\vec{I}/\partial t = 0$	**Integrable** $\hat{H}_0 = \sum_{\mu} E_{\mu} \lvert\mu\rangle\langle\mu\rvert, \quad \lvert\mu\rangle = \lvert\vec{I}\rangle$
Perturbation $V; \quad \partial\vec{I}/\partial t \neq 0$	**Perturbation** $\hat{V} = \sum_{\mu,\nu} V_{\mu,\nu} \lvert\mu\rangle\langle\nu\rvert$
KAM	**Localized**
Ergodic (chaotic)	**Extended ?**

Question:

What is the reason to speak about localization if we in general do not know the space in which the system is localized ?

Need an invariant (basis independent) criterion of the localization