Highly entangled states: A totally new kind of quantum materials and an unification of light and electrons

Xiao-Gang Wen, Perimeter/MIT, Dec. 2012

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A totally new kind of quantum materials
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Postdoc at Institute for Advanced Study, Tsinghua University

2-year postdoctoral Fellows at $20,000/year

3-year C. N. Yang Junior Fellows at $30,000/year

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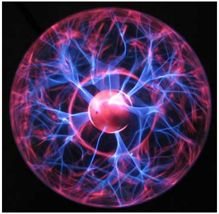
In primary school, we learned ...

there are four states of matter:

Solid

Liquid

Gas



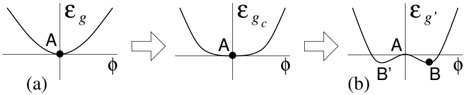
Plasma

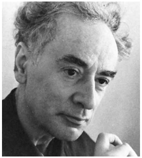
In university, we learned ...

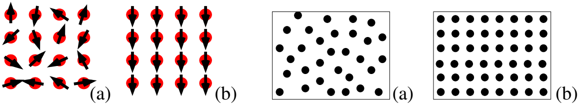
there are much more than four phases:

different phases = different symmetry breaking

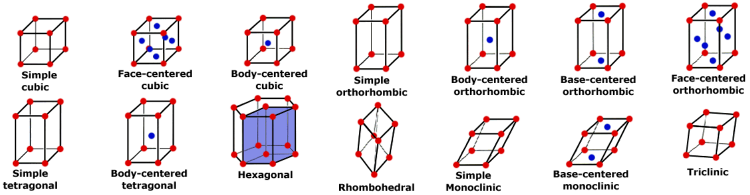
→ Landau symmetry breaking theory





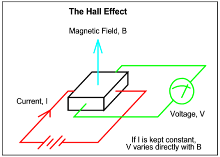


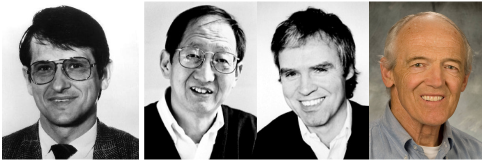
Group theory: From 230 ways of translation symmetry breaking, we obtain the 230 crystal orders in 3D



In graduate study after 1980's, we learned ...

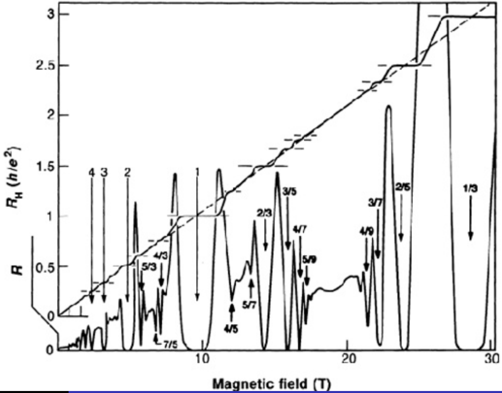
there are phases beyond symmetry-breaking:





$E_y = R_H j_x, \ \ R_H = \frac{p}{q} \frac{h}{e^2}$

- 2D electron gas in magnetic field has many **quantum Hall (QH) states**



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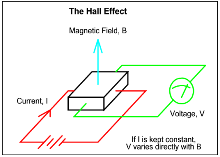
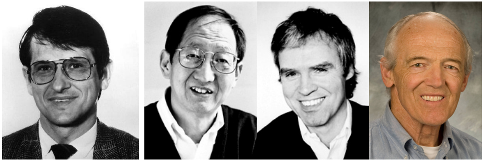
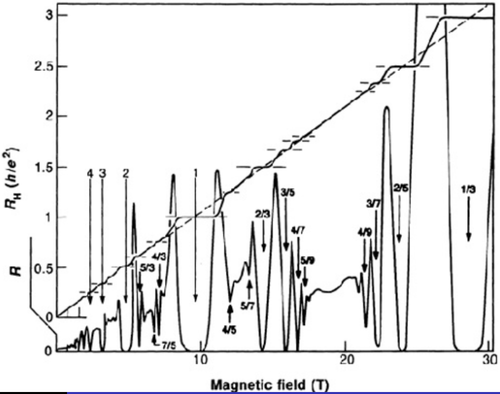




$E_y = R_H j_x, \ \ R_H = \frac{p}{q}\frac{h}{e^2}$

- 2D electron gas in magnetic field has many **quantum Hall (QH) states** that all have the **same symmetry**.
- Different QH states cannot be described by symmetry breaking theory.
- We call the new order **topological order** Wen 89



What is topological order?

- Topological order is a property or a *pattern* in the ground state wave function

$$\Phi(x_1, x_2, ..., x_N), \qquad N \sim 10^{10} - 10^{23}$$

But how to see a pattern in a wave function that we cannot even write down?

- Symmetry breaking order is also a *pattern* in the ground state wave function, where we examin if the wave function is invariant under symmetry operation U or not:

$$U[\Phi(x_1, x_2, ..., x_N)] =?= \Phi(x_1, x_2, ..., x_N)$$

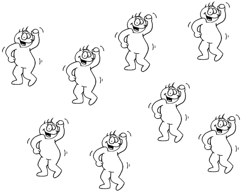
$\rightarrow$ pattern of symmetry breaking.

- *Use dancing picture to understand the pattern of topological order and pattern of symmetry breaking order.*

Symmetry breaking orders through pictures

Ferromagnet

Anti-ferromagnet



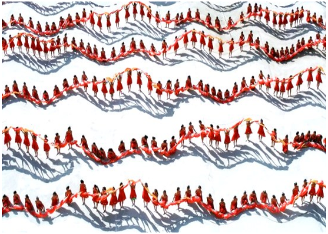
Superfluid of bosons

Superconductor of fermions

- every spin/particle is doing its own dancing,
  every spin/particle is doing the same dancing → **Long-range order**

Topological orders through pictures

FQH state



String liquid (spin liquid)

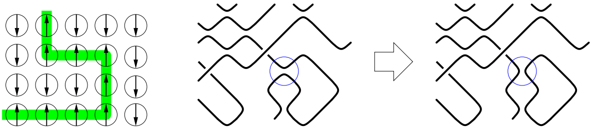
- **Global dance**:
  All spins/particles dance following a local dancing "rules"
  → The spins/particles dance collectively
  → a global dancing pattern.

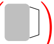
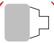
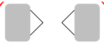
Local dancing rule $\to$ global dancing pattern

- Local dancing rules of a FQH liquid:
  (1) every electron dances around clock-wise
  (Φ_{FQH} only depends on $z = x + \text{i}y$)
  (2) takes exactly three steps to go around any others
  (Φ_{FQH}'s phase change 6π)
  $\to$ Global dancing pattern $\Phi_{\text{FQH}}(\{z_1, ..., z_N\}) = \prod(z_i - z_j)^3$

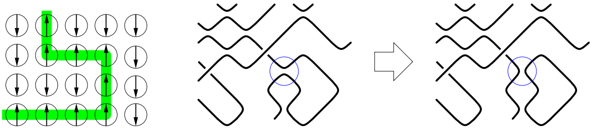
- Local dancing rules are enforce by the Hamiltonian
  to lower the ground state energy.

Local dancing rule → global dancing pattern



- Local dancing rules of a string liquid:
  (1) Dance while holding hands (no open ends)
  (2) Φ_{str} () $=$ Φ_{str} () , Φ_{str} () $=$ Φ_{str} ()
  → Global dancing pattern Φ_{str} () $= 1$

Local dancing rule → global dancing pattern



- Local dancing rules of a string liquid:
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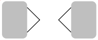
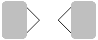
  (2) Φ_{str} () $=$ Φ_{str} (), Φ_{str} () $=$ Φ_{str} ()
  → Global dancing pattern Φ_{str} () $= 1$

- Local dancing rules of another string liquid:
  (1) Dance while holding hands (no open ends)
  (2) Φ_{str} () $=$ Φ_{str} (), Φ_{str} () $= -$ Φ_{str} ()
  → Global dancing pattern Φ_{str} () $= (-)^{\#\text{ of loops}}$

- Two string-net condensations → two topological orders Levin-Wen 05

What is the significance of topological order?

Globel danicing pattern is a nice picture for topological order.
But does it mean anything?
Does topological order have any experiemntal significance?
Does topological order have any new experiemntal properties, that is different from any symmetry breaking order?

How to measure/study topological order in experiemnts?

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Does topological order have any new experiemntal properties, that is different from any symmetry breaking order?

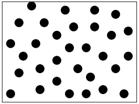
How to measure/study topological order in experiemnts?

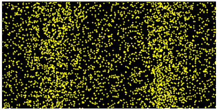
- Topological orders produce **new kind of waves** (collective excitations above the topo. ordered ground states).
  → *change our view of universe*
- The defects of topological order carry **fractional statistics** (including non-Abelian statistics) and **fractional charges** (if there is symmetry).
  → *a medium for topological quantum memory and computations.*
- Some topological orders have topologically protected **gapless boundary excitations**
  → *perfect conducting surfaces despite the insulating bulk.*

Principle of emergence: from order to physical properties

Different orders → different wave equations for the deformations of order → different physical properties.

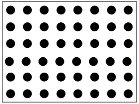
- Atoms in superfluid have a random distribution
  → cannot resist shear deformations (which do nothing)
  → liquids do not have shapes



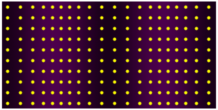


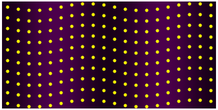
Wave Eq. → Euler Eq.
$\partial_t^2 \rho - \partial_x^2 \rho = 0$ One longitudinal mode

- Atoms in solid have a ordered lattice distribution
  → can resist shear deformations
  → solids have shapes



Wave Eq. → elastic Eq.
$\partial_t^2 u^i - C^{ijkl} \partial_{x^j} \partial_{x^k} u^l = 0$



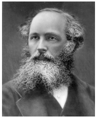


One longitudinal mode and two transverse modes

Origin of photons, gluons, electrons, quarks, etc

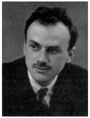
- Do all waves and wave equations emerge from some orders?

Wave equations for elementary particles



- Maxwell equation → Photons
  $\partial \times \mathbf{E} + \partial_t \mathbf{B} = \partial \times \mathbf{B} - \partial_t \mathbf{E} = \partial \cdot \mathbf{E} = \partial \cdot \mathbf{B} = 0$

- Yang-Mills equation → Gluons
  $\partial^\mu F^a_{\mu\nu} + f^{abc} A^{\mu b} F^c_{\mu\nu} = 0$



- Dirac equation → Electrons/quarks
  $[\partial_\mu \gamma^\mu + m]\psi = 0$

What orders produce the above waves? What are the origins of light (gauge bosons) and electrons (fermions)?

Elementary or emergent?

- But none of the symmetry breaking orders can produce:
  - electromagnetic wave satisfying the Maxwell equation
  - gluon wave satisfying the Yang-Mills equation
  - electron wave satisfying the Dirac equation.

Elementary or emergent?

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- We have seen that topological order corresponds to to new states of quantum matter, such as FQH states and spin-liquid states.
- Can topological order produces the Maxwell equation, Yang-Mills equation, and the Dirac equation?

Elementary or emergent?

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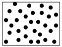
- We have seen that topological order corresponds to to new states of quantum matter, such as FQH states and spin-liquid states.

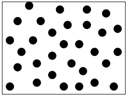
- Can topological order produces the Maxwell equation, Yang-Mills equation, and the Dirac equation?

Yes

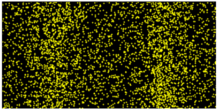
- A particular topologically ordered state, string-net liquid, provide a unified origin of light, electrons, quarks, gluons, ...

Topological order (closed strings) → emergence of electromagnetic waves (photons)

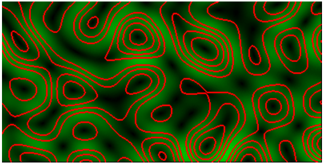
- Wave in superfluid state $|\Phi_{SF}\rangle = \sum_{\text{all position conf.}}$ $\Big|$  $\Big\rangle$:

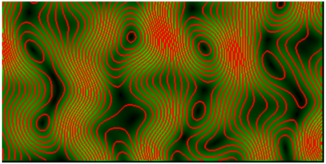


density fluctuations:
Euler eq.: $\partial_t^2 \rho - \partial_x^2 \rho = 0$
→ Longitudinal wave



- Wave in closed-string liquid $|\Phi_{string}\rangle = \sum_{\text{closed strings}}$ $\Big|$  $\Big\rangle$:

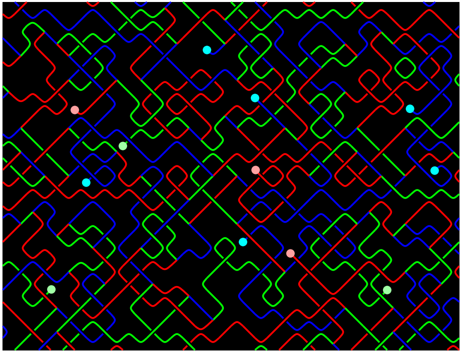




String density $\mathbf{E}(\mathbf{x})$ fluctuations → waves in string condensed state.
Strings have no ends → $\partial \cdot \mathbf{E} = 0$ → **only two transverse modes**.
Equation of motion for string density → Maxwell equation:
$\dot{\mathbf{E}} - \partial \times \mathbf{B} = \dot{\mathbf{B}} + \partial \times \mathbf{E} = \partial \cdot \mathbf{B} = \partial \cdot \mathbf{E} = 0$. ($\mathbf{E}$ electric field)

Topological order (string nets)→ Emergence of Yang-Mills theory (gluons)

- If string has different types and can branch
  → string-net liquid → Yang-Mills theory
- Different ways that strings join → different gauge groups

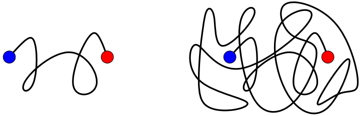


A picture of our vacuum

A string–net theory of light and electrons

Closed strings → Maxwell gauge theory
String-nets → Yang-Mills gauge theory

Topological order →
Emergence of electrons



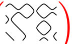
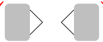
- In string condensed states, the ends of string be have like point particles
  - with quantized (gauge) charges
  - with Fermi statistics

Levin-Wen 2003

- **String-net/topological-order provides a way to unify gauge interactions and Fermi statistics in 3D**

Emergence of fractional spin/statistics

- Why electron carry spin-1/2 and Fermi statistics?

- $\Phi_{\text{str}}\Big($  $\Big) = 1$ string liquid $\Phi_{\text{str}}\Big($  $\Big) = \Phi_{\text{str}}\Big($  $\Big)$

  360° rotation:  $\to$  and  $=$  $\to$ 

   $+$  has a spin 0 mod 1.  $-$  has a spin $1/2$ mod 1.

Emergence of fractional spin/statistics

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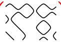
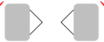
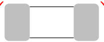
- Φ_{str} () $= 1$ string liquid Φ_{str} () $= \Phi_{\text{str}}$ ()

  360° rotation:  $\to$  and  $=$  $\to$ 

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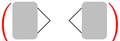
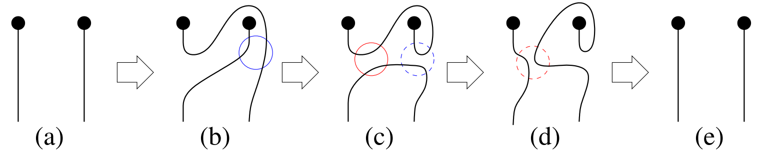
- Φ_{str} () $= (-)^{\#\text{ of loops}}$ string liquid Φ_{str} () $= -\Phi_{\text{str}}$ ()

  360° rotation:  $\to$  and  $= -$  $\to -$ 

   $+ \mathrm{i}$  has a spin $-1/4$ mod 1.  $- \mathrm{i}$  has a spin $1/4$ mod 1.

Spin-statistics theorem



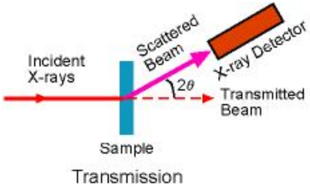
- (a) → (b) = exchange two string-ends.
- (d) → (e) = 360° rotation of a string-end.
- Amplitude (a) = Amplitude (e)
- Exchange two string-ends plus a 360° rotation of one of the string-end generate no phase.

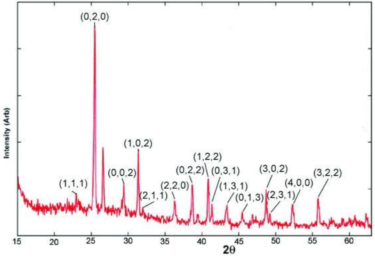
→ **Spin-statistics theorem**

To define a physical concept, such as symmetry-breaking order or topological order, is to design a probe to measure it

For example,

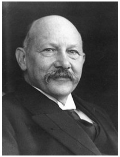
- crystal order is defined/probed by X-ray diffraction:

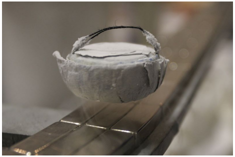




Symmetry-breaking orders through experiments

Order	Experiment
Crystal order	X-ray diffraction
Ferromagnetic order	Magnetization
Anti-ferromagnetic order	Neutron scattering
Superconducting order	Zero-resistance & Meissner effect
Topological order	???



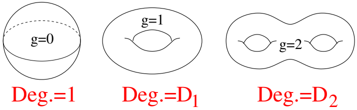


- All the above probes are linear responses. But topological order cannot be probed/defined through linear responses.

Topological orders through experiments (1990)

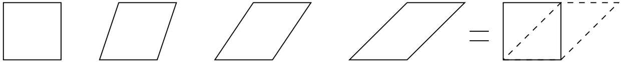
Topological order can be defined "experimentally" through two unusual topological probes (at least in 2D)

(1) **Topology-dependent ground state degeneracy** D_g Wen 89



(2) **Non-Abelian geometric's phases** of the degenerate ground state from deforming the torus: Wen 90

- Shear deformation T: $|\Psi_\alpha\rangle \to |\Psi'_\alpha\rangle = T_{\alpha\beta}|\Psi_\beta\rangle$



- 90° rotation S: $|\Psi_\alpha\rangle \to |\Psi''_\alpha\rangle = S_{\alpha\beta}|\Psi_\beta\rangle$

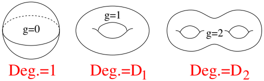
- T, S, define topological order "experimentally".
- T, S is a *universal probe* for any 2D topological orders, just like X-ray is a universal probe for any crystal orders.

Symmetry-breaking/topological orders through experiments

Order	Experiment
Crystal order	X-ray diffraction
Ferromagnetic order	Magnetization
Anti-ferromagnetic order	Neutron scattering
Superconducting order	Zero-resistance & Meissner effect
Topological order (Global dancing pattern)	Topological degeneracy, non-Abelian geometric phase

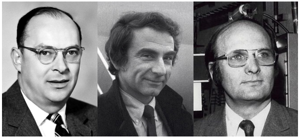
- The linear-response probe **Zero-resistance** and **Meissner effect** define **superconducting order**. Treating the EM fields as non-dynamical fields
- The topological probe **Topological degeneracy** and **non-Abelian geometric phases** T, S define a completely new class of order – **topologically order**.
- T, S determines the quasiparticle statistics. Keski-Vakkuri & Wen 93; Zhang-Grover-Turner-Oshikawa-Vishwanath 12; Cincio-Vidal 12

What is the microscopic picture of topological order?



represent an experimental definition of topological order.

- But what is the microscopic understanding of topological order?
- Zero-resistance and Meissner effect → experimental definition of superconducting order.
- It took 40 years to gain a microscopic picture of superconducting order: **electron-pair condensation**



Bardeen-Cooper-Schrieffer 57

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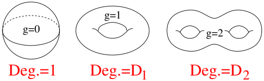

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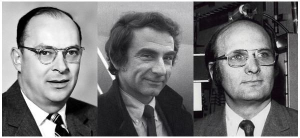

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- It took 20 years to gain a microscopic understanding of topological order: **long-range entanglements** Chen-Gu-Wen 10 (defined by local unitary trans. and motivated by topological entanglement entropy). Kitaev-Preskill 06,Levin-Wen 06

Quantum entanglements through examples

- $|\uparrow\rangle \otimes |\downarrow\rangle$ = direct-product state $\rightarrow$ unentangled (classical)

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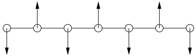
-  $= |\downarrow\rangle \otimes |\uparrow\rangle \otimes |\downarrow\rangle \otimes |\uparrow\rangle \otimes |\downarrow\rangle ...$ $\to$ unentangled

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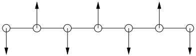
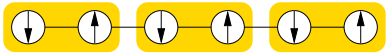
-  $= |\downarrow\rangle \otimes |\uparrow\rangle \otimes |\downarrow\rangle \otimes |\uparrow\rangle \otimes |\downarrow\rangle ...$ $\to$ unentangled
-  $= (|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle) \otimes (|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle) \otimes ...$ $\to$
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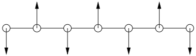
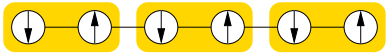
-  $= |\downarrow\rangle \otimes |\uparrow\rangle \otimes |\downarrow\rangle \otimes |\uparrow\rangle \otimes |\downarrow\rangle \ldots$ → unentangled
-  $= (|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle) \otimes (|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle) \otimes \ldots$ →
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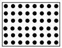
- Crystal order: $|\Phi_{\text{crystal}}\rangle = \Big|$  $\Big\rangle = |0\rangle_{x_1} \otimes |1\rangle_{x_2} \otimes |0\rangle_{x_3} \cdots$
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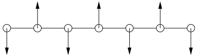
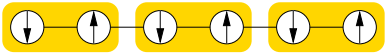
-  $= |\downarrow\rangle \otimes |\uparrow\rangle \otimes |\downarrow\rangle \otimes |\uparrow\rangle \otimes |\downarrow\rangle \dots$ → unentangled
-  $= (|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle) \otimes (|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle) \otimes \dots$ →
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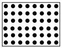
- Crystal order: $|\Phi_{\text{crystal}}\rangle = \Big|$  $\Big\rangle = |0\rangle_{x_1} \otimes |1\rangle_{x_2} \otimes |0\rangle_{x_3} \cdots$
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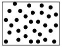
  $|\Phi_{SF}\rangle = \sum_{\text{all conf.}} \Big|$  $\Big\rangle$

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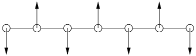
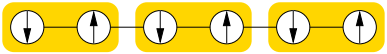
-  $= |\downarrow\rangle \otimes |\uparrow\rangle \otimes |\downarrow\rangle \otimes |\uparrow\rangle \otimes |\downarrow\rangle ...$ → unentangled
-  $= (|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle) \otimes (|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle) \otimes ...$ →
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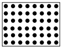
- Crystal order: $|\Phi_{\text{crystal}}\rangle = \Big|$  $\Big\rangle = |0\rangle_{x_1} \otimes |1\rangle_{x_2} \otimes |0\rangle_{x_3} \cdots$
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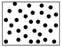
  $|\Phi_{\text{SF}}\rangle = \sum_{\text{all conf.}} \Big|$  $\Big\rangle = (|0\rangle_{x_1} + |1\rangle_{x_1} + ..) \otimes (|0\rangle_{x_2} + |1\rangle_{x_2} + ..) ...$
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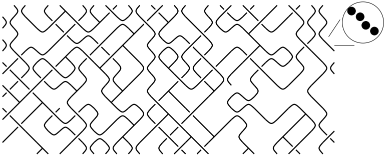
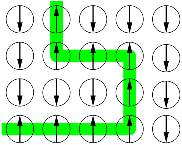
How to make long range entanglements (topo. orders)

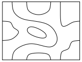
To make topological order, we need to sum over different product states, but we should not sum over everything.

- Sum over a subset of the particle configurations, by first join the particles into strings, then sum over the loop states





→ string-net condensation (string liquid): Levin-Wen 05

$|\Phi_{\text{loop}}\rangle = \sum_{\text{all loop conf.}}$ $\Big|$  $\Big\rangle$ which is not a direct-product state and not a local deformation of direct-product states
→ non-trivial **topological orders** (long-range entanglements)

Pattern of long-range entanglements = topological order

For gapped systems with no symmetry:

- According to Landau theory, no symmetry to break
  $\to$ all systems belong to one trivial phase

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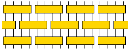
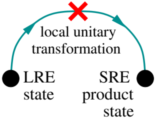
  - There are **long range entangled (LRE) states**
  - There are **short range entangled (SRE) states**

$|\text{LRE}\rangle \neq$  $|\text{product state}\rangle = |\text{SRE}\rangle$



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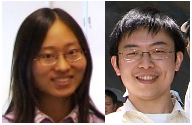

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  - There are **long range entangled (LRE) states** $\to$ many phases
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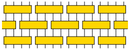
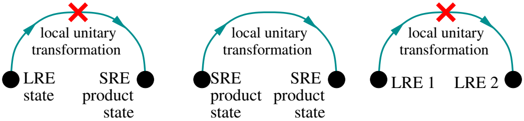
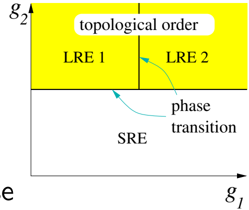
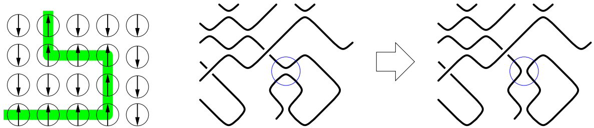
$|\text{LRE}\rangle \neq$  $|\text{product state}\rangle = |\text{SRE}\rangle$

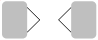


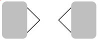


- All SRE states belong to the same trivial phase
- LRE states can belong to many different phases
  = different **patterns of long-range entanglements** defined by the LU trans.
  = different **topological orders**
  $\to$ A classification by **tensor category theory** Levin-Wen 05, Chen-Gu-Wen 2010

Local dancing rule → global dancing pattern



- Local dancing rules of a string liquid:
  (1) Dance while holding hands (no open ends)
  (2) Φ_{str} () $=\Phi_{\text{str}}$ (), Φ_{str} () $=\Phi_{\text{str}}$ ()
  → Global dancing pattern Φ_{str} () $=1$

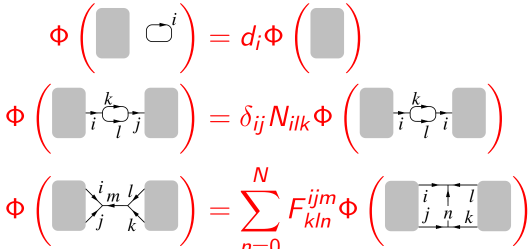
- Local dancing rules of another string liquid:
  (1) Dance while holding hands (no open ends)
  (2) Φ_{str} () $=\Phi_{\text{str}}$ (), Φ_{str} () $=-\Phi_{\text{str}}$ ()
  → Global dancing pattern Φ_{str} () $=(-)^{\#\text{ of loops}}$

- Two string-net condensations → two topological orders Levin-Wen 05

More general dancing rules → Tensor category theory

The local dancing rules can be described by data $d_i, N_{ijk}, F^{ijm}_{kln}$:

Levin-Wen 05



which must satisfy $F^{ijk}_{j^* i^* 0} = \frac{v_k}{v_i v_j} N_{ijk}, \quad v_i^2 = d_i$

$$F^{ijm}_{kln} = F^{lkm^*}_{jin} = F^{jim}_{lkn^*} = F^{imj}_{k^* nl} \frac{v_m v_n}{v_j v_l}$$

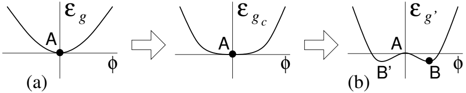
$$\sum_{n=0}^{N} F^{mlq}_{kpn} F^{jip}_{mns} F^{jsn}_{lkr} = F^{jip}_{qkr} F^{riq}_{mls}$$

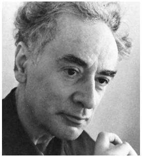
The theory about the solutions = tensor category theory

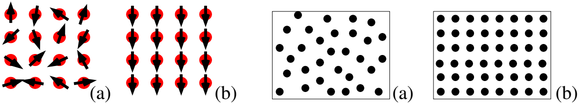
→ classify 2D gapped phases with no symmetry (topological order)

Short-range entanglements that break symmetry
$\rightarrow$ Landau symmetry breaking phases

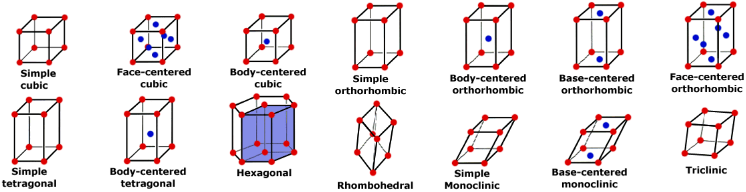
different phases = different symmetry breaking







From 230 ways of translation symmetry breaking, we obtain the 230 crystal orders in 3D



Short-range entanglements w/ symmetry $\rightarrow$ SPT phases

For gapped systems with a symmetry G (no symmetry breaking):

- there are **LRE symmetric states** $\rightarrow$ many different phases
- there are **SRE symmetric states** $\rightarrow$ one phase

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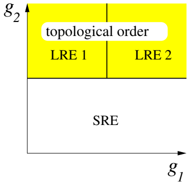

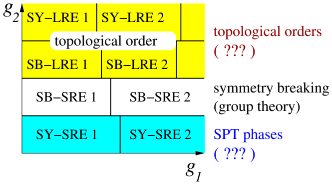

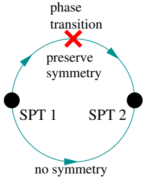

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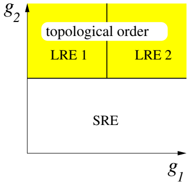
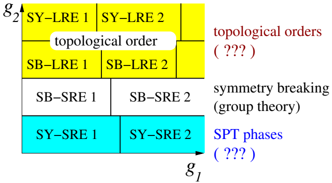
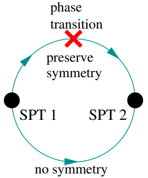
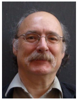






- Haldane phase of 1D spin-1 chain w/ $SO(3)$ symm. Haldane 83

1D 

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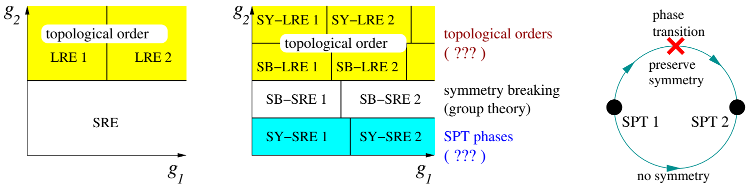
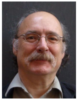
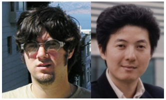
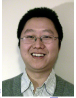
or **symmetry protected topological (SPT)** phase



- Haldane phase of 1D spin-1 chain w/ $SO(3)$ symm. Haldane 83
- Topo. insulators w/ $U(1) \times T$ symm.: 2D Kane-Mele 05; Bernevig-Zhang 06 and 3D Moore-Balents 07; Fu-Kane-Mele 07

1D  2D   3D  

Compare topological order and topological insulator

- **Topological order** describes states with *long-range entanglements*
  - The essence: long-range entanglements
- **Topological insulator** is a state with *short-range entanglements, particle number conservation, and time reversal symmetry*, which is an example of SPT phases.
  - The essence: symmetry entangled with short-range entanglements

	Topo. order	Topo. Ins.	Band Ins.
Entanglements	long range	short range	short range
Fractional charges of finite-energy defects	Yes	No	No
Fractional statistics of finite-energy defects	Yes	No	No
Proj. non-Abelian stat. of infinite-energy defects	Yes $>$ Majorana	Yes only Majorana	Yes only Majorana
Gapless boundary	topo. protected	symm. protected	not protected

Highly entangled quantum matter: A new chapter of condensed matter physics

Quantum States of Matter (gapped)

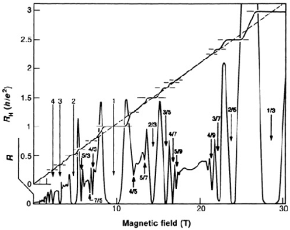
- Long–Rang Entangled = Topological order — Tensor category theory
  - FQH State
  - Real Super–conductor
  - Spin Liquid

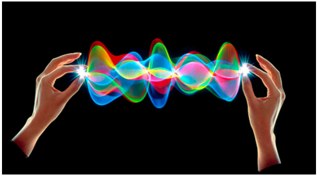
  Gauge theory, Fractional statistics, Fermi statistics, Boundary excitations, Topological degeneracy

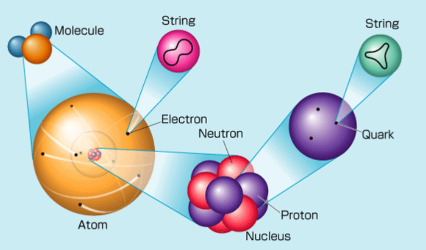
- Short–Range Entangled
  - Symm. Breaking — Group Theory
    - Crystal
    - Magnet
    - Liquid Crystal

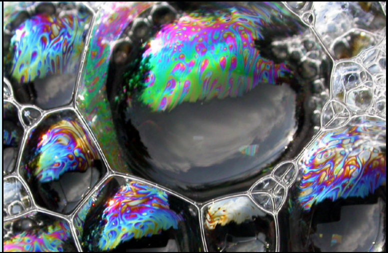
    Shape, Insulator, Information storage, Gapless Goldstone mode
  - With symmetry — Group cohomology theory
    - Haldane Phase
    - Topological Insulator

    Boundary excitations, Anti–localization









Topological order
is the source of many wonders