Anderson localization of a Majorana fermion

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arXiv:0901.1914,
arXiv:1111.0339,
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arXiv:1307.0372

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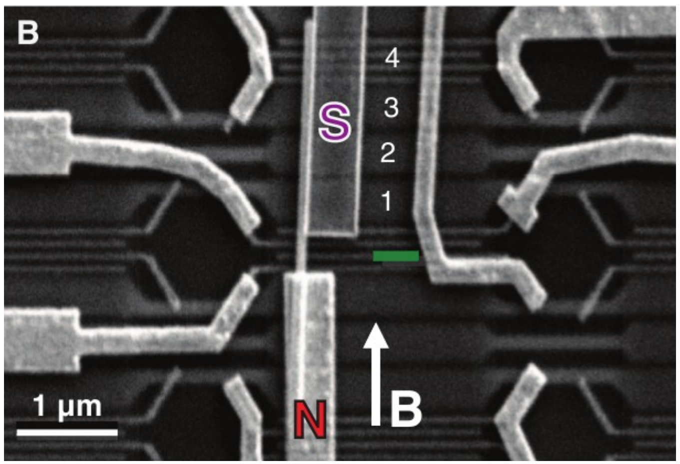
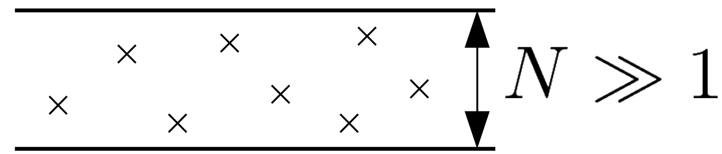
Motivation

1. Experimental search for Majorana fermions in solid-state devices



from V.Mourik et al, Science 336, 1003 (2012)
first report of a Majorana fermion:
InSb wire (spin-orbit) + magnetic field
+ proximity-induced superconductivity

2. Recent theoretical progress on Anderson localization in quantum wires in the unitary symmetry class (using Efetov's nonlinear supersymmetric sigma model)



M.Skvortsov, P.Ostrovsky,
JETP Lett. 85, 72 (2007);
D.I., P.Ostrovsky, M.Skvortsov,
PRB 79, 205108 (2009).

How would a Majorana fermion localize in a disordered wire?

Plan of the talk

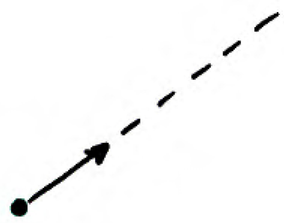
1. Anderson localization in a quasi-1D disordered wire (in the unitary symmetry class: broken time-reversal symmetry)

2. Interpretation in terms of Mott hybridization

3. Application to Normal metal – Superconductor junctions (including topological superconductors with Majorana fermions)

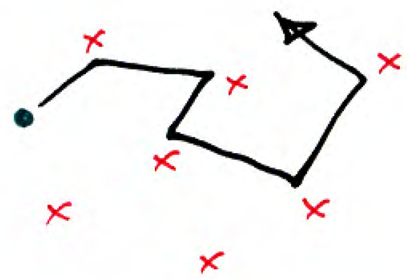
Anderson localization: introduction

1. Free particle:



2. Classical diffusion:

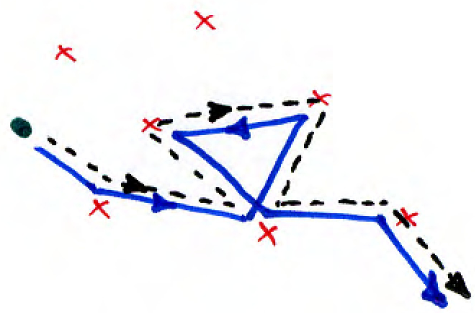
$L^2 \propto t$



3. Quantum interference:

$$|A_1 + A_2|^2 = |A_1|^2 + |A_2|^2 + 2\mathrm{Re}\, A_1^* A_2$$

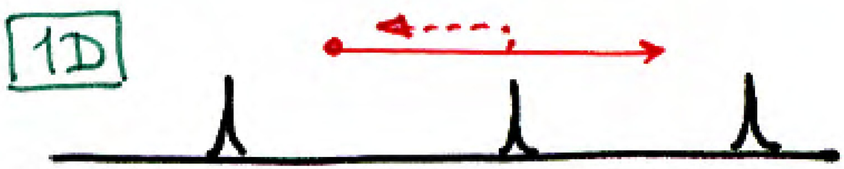
4. Localization corrections:



In 1 and 2 dimensions, interference suppresses the diffusion completely at arbitrary strength of disorder: the particle stays in a finite region of space (localization) [Mott, Twose '61; Berezinsky '73; Abrahams, Anderson, Licciardello, Ramakrishnan '79]

One-dimensional models

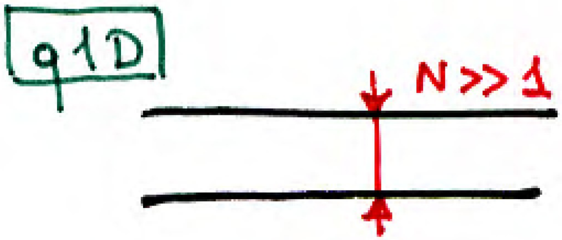
Particle on a line (strictly 1D):



[Berezinsky technique: equations on the probability distribution of the scattering phase (exact results available)]

$$\xi \sim l$$

Thick wire (quasi 1D):



[Efetov's supersymmetric nonlinear sigma model]

$$\xi \sim Nl$$

ξ – localization length

l – mean free path

Rescaled to the localization length ξ, localization looks similar in the two models. Which properties are universal?

Quantitative description of localization

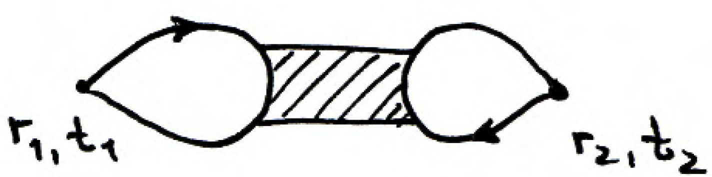
In the normal wire, localization is not visible in the average of a single Green's function:

$\langle G(r) \rangle$ decays at the length scale of a mean free path

Averaging two Green's functions (TWO types of averages):

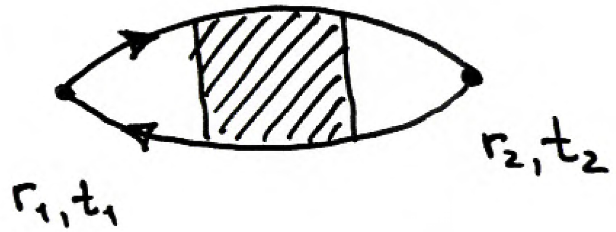
1. $\langle G(1,1) G(2,2) \rangle$

(correlations of local density of states)



2. $\langle G(1,2) G(2,1) \rangle$

(dynamic response function)



Correlation functions: formal definitions

1. LDOS correlations:

$$R(\omega, x) = \nu^{-2} \left\langle \sum_{n,m} |\Psi_n(0)|^2 \, |\Psi_m(x)|^2 \, \delta(E_n - E_m - \omega) \, \delta(E - E_n) \right\rangle$$

2. Dynamic response function:

$$S(\omega, x) = \nu^{-2} \left\langle \sum_{n,m} \Psi_n^*(0) \Psi_n(x) \Psi_m^*(x) \Psi_m(0) \times \delta(E_n - E_m - \omega) \, \delta(E - E_n) \right\rangle$$

(averaging is over disorder realizations)

- length unit: localization length ξ
- energy unit: Δ_ξ = level spacing at length ξ
  = Thouless energy at length ξ

$$\Delta_\xi = D/\xi^2 \sim (\nu_1 \xi)^{-1}$$

D – diffusion constant
ν_1 – 1D density of states

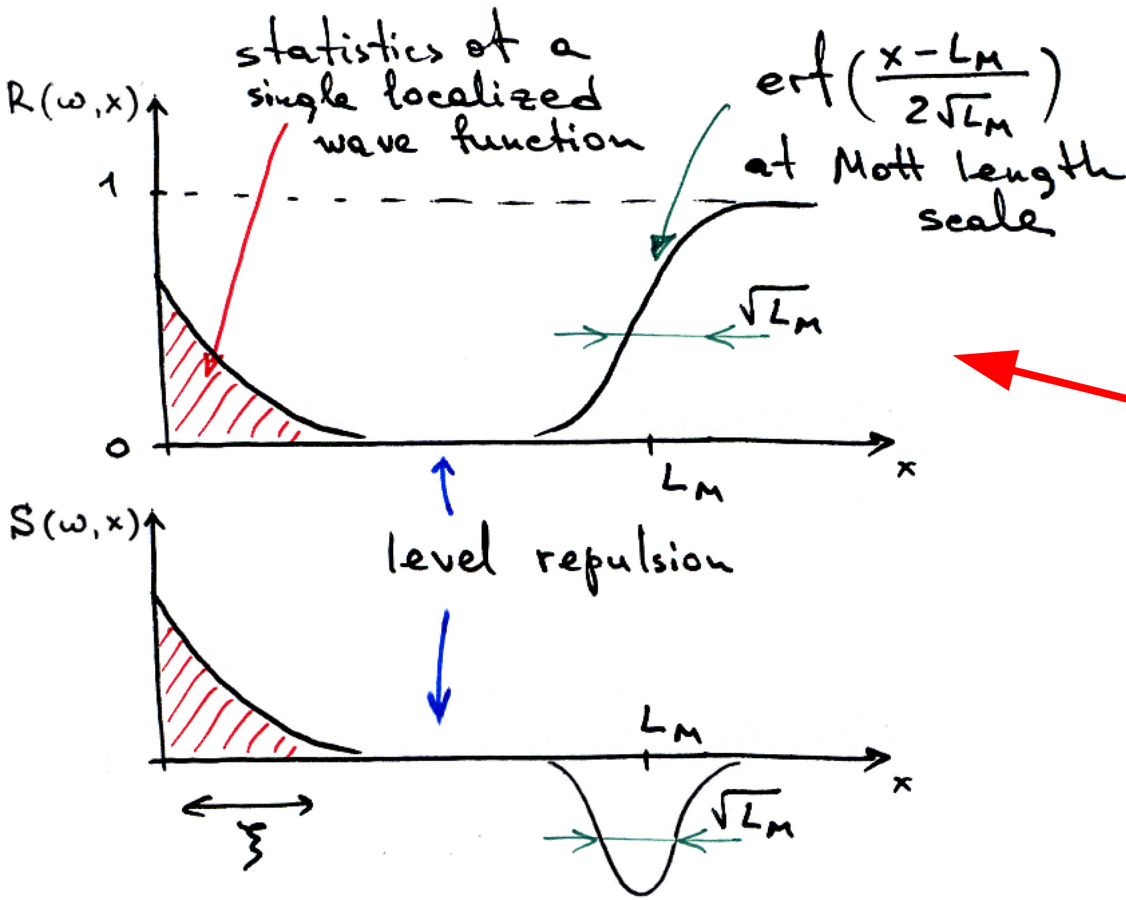
Available analytical results

<table>
<tr><th></th><th>single-wave-function statitics</th><th>$R(\omega, x)$</th><th>$S(\omega, x)$</th></tr>
<tr><td>strictly 1D (S1D)</td><td rowspan="3">universality of statistics [Gogolin '76, Kolokolov '95, Mirlin '00]</td><td colspan="2">[Gor'kov, Dorokhov, Prigara '83]</td></tr>
<tr><td>quasi-1D unitary (Q1D-U): broken time-reversal symmetry</td><td>this work</td><td>???</td></tr>
<tr><td>quasi-1D orthogonal (Q1D-O): preserved time-reversal symmetry</td><td>???</td><td>???</td></tr>
</table>

(assuming Gaussian white-noise disorder and quasiclassical regime $kl \gg 1$)

Structure of correlations in 1D

[Gor'kov, Dorokhov, Prigara '83] $\omega \ll \Delta_\xi$

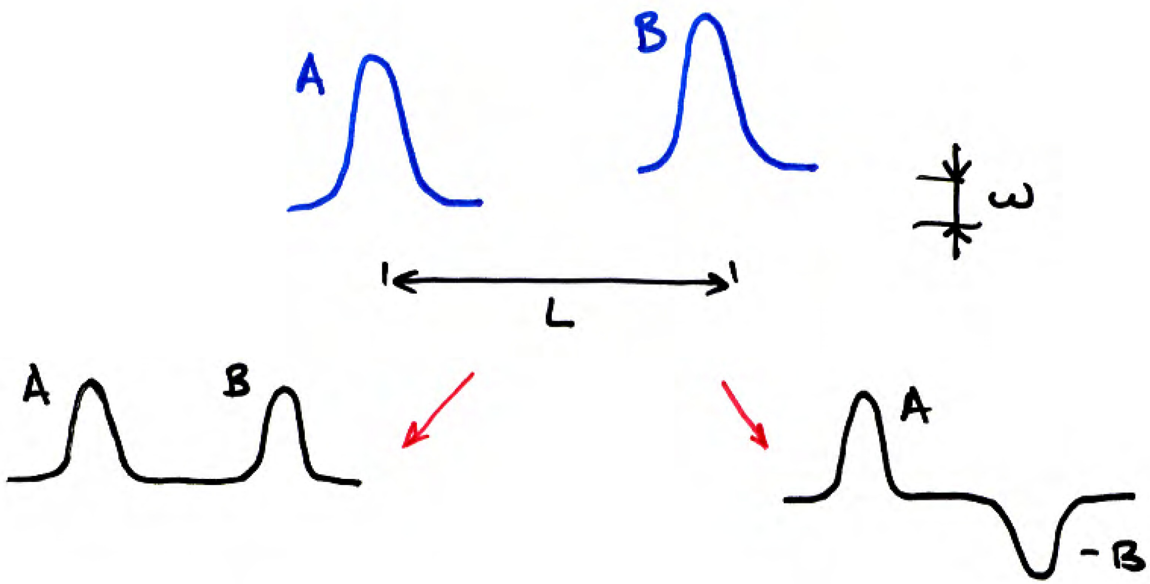


will be compared with our sigma-model results in Q1D

qualitatively explained by Mott hybridization argument [Mott '70]

$L_M \sim \log(\Delta_\xi/\omega)$ – Mott length scale

Mott argument (wave function hybridization)



1. At short distances ($x \leq \xi$), the two eigenfunctions have the same profile (single localized wave function)

2. Hybridization is important as long as the splitting

$$\Delta_\xi \exp(-L/2\xi) > \omega \quad \Leftrightarrow \quad L < L_M = 2\xi \ln(\Delta_\xi/\omega)$$

Details of the sigma-model calculations: action

Averaging over disorder $\Rightarrow$ Nonlinear supersymmetric sigma model [Efetov, '83]

For simplicity, we consider the unitary symmetry class (time-reversal symmetry completely broken: e.g., by a magnetic field).

$$Z = \int [DQ]\, e^{-S}, \qquad S = -\frac{1}{4}\,\mathrm{STr} \int dx \left[\frac{1}{2} \left(\frac{dQ}{dx} \right)^2 + i\omega \Lambda Q \right]$$

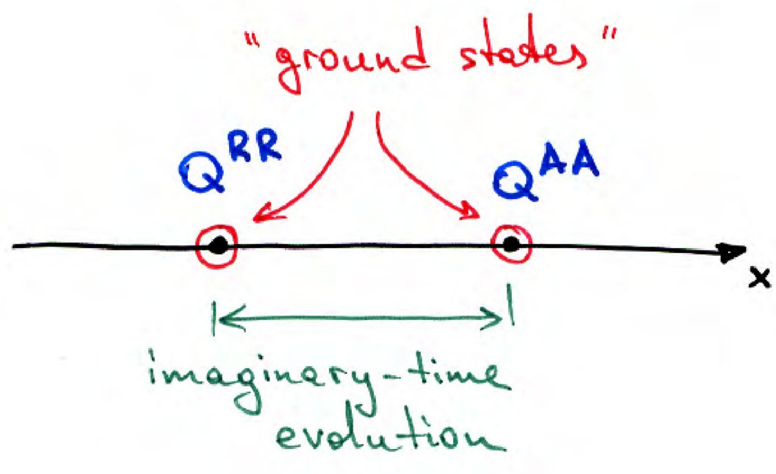
x and ω in the units of ξ and Δ_ξ, respectively
Q is a 4×4 supermatrix with constraint $Q^2 = 1$ (from $N \gg 1$),
fermion-boson (FB) and retarded-advanced (RA) sectors

$$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{RA}$$

Details of the sigma-model calculations: transfer-matrix formalism

Relation to the correlations of the density of states:

$$R(\omega, x) \equiv \nu^{-2} \langle \rho_E(0) \rho_{E+\omega}(x) \rangle = \frac{1}{2} \left[1 - \mathrm{Re} \langle Q^{RR}_{BB}(0) Q^{AA}_{BB}(x) \rangle \right]$$



$$R(\omega, x) = 1 + \frac{1}{2} \mathrm{Re} \langle \Psi_0 | e^{-Hx} | \Psi_0 \rangle$$

$\Psi_0(\lambda_B, \lambda_F)$ – known ground state (in terms of Bessel functions) [Skvortsov, Ostrovsky '06, D.I., Skvortsov, '08]

Details of the sigma-model calculations: separation of variables

Luckily, the variables in the Hamiltonian separate

$$\lambda_B \in [1, +\infty)\,, \qquad \lambda_F \in [-1, 1]$$

$$H = H_B + H_F$$

$$H_B = -\partial_{\lambda_B}(\lambda_B^2 - 1)\partial_{\lambda_B} + \Omega\lambda_B$$
$$H_F = -\partial_{\lambda_F}(1 - \lambda_F^2)\partial_{\lambda_F} - \Omega\lambda_F$$

where $\Omega = -i\omega/2$.

Fermionic part: compact, can be solved perturbatively in ω

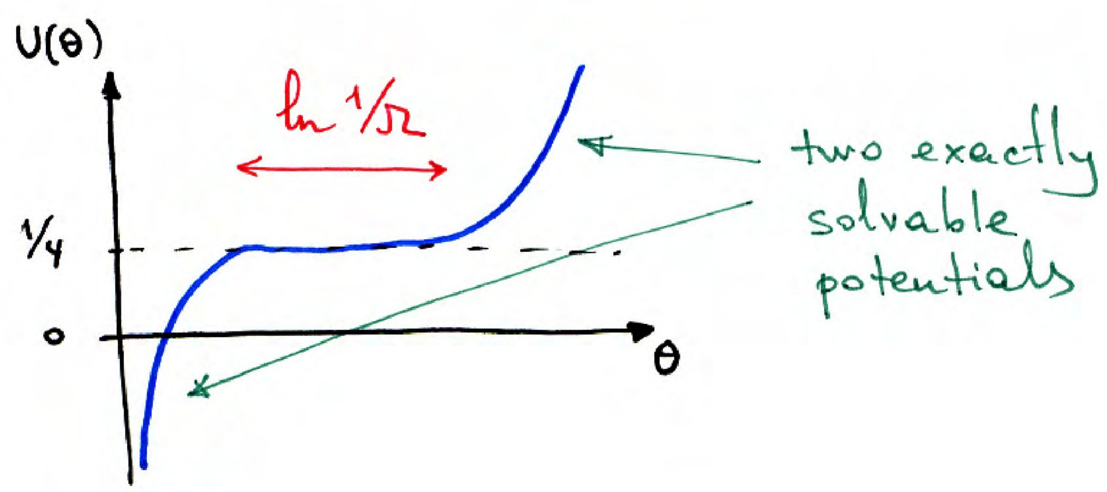
Bosonic part: non-compact, expansion contains both powers and logarithms of ω

For calculation, we assume Ω real positive, then analytically continue

Details of the sigma-model calculations: matching Legendre and Bessel asymptotics in the bosonic sector

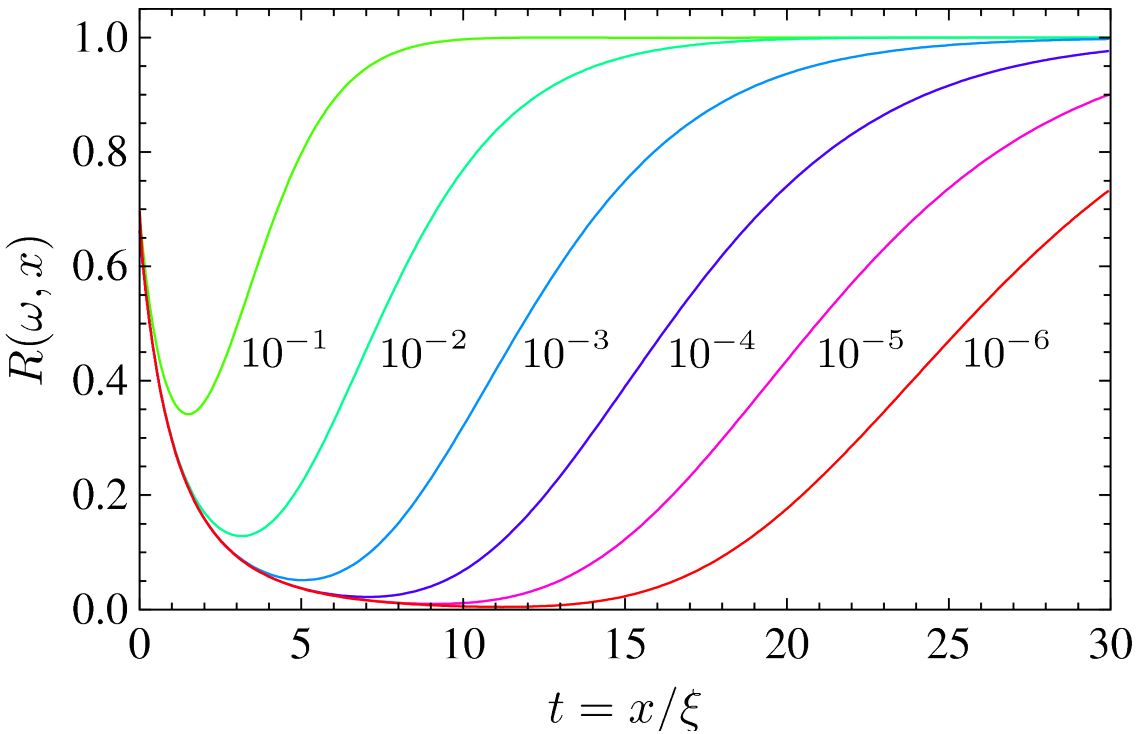
If one "unfolds" the λ_B axis ($\lambda_B = \cosh\theta$)

$$H_\theta = -\frac{d^2}{d\theta^2} + U(\theta)\,, \qquad U(\theta) = \frac{1}{4} - \frac{1}{4\sinh^2\theta} + \Omega\cosh\theta$$



Eigenstates may be constructed order by order in Ω by matching the asymptotics of Legendre (at small θ) and modified Bessel (at large θ) functions (technical part)

Results of the sigma-model calculations



The leading asymptotics is the same as in S1D: single-wave-function correlations at small x and $\mathrm{erf}(...)$ at large x.

Subleading terms in ω are different. At $x \leq \xi$

$$R(x, \omega) = R_0 + O(\omega^2 \ln^2 \omega) \qquad \text{in Q1D-U}$$

$$R(x, \omega) = R_0 + O(\omega^2 \ln \omega) \qquad \text{in S1D}$$

Universality in two-point correlations

Sigma-model calculations for $R(\omega, x)$ in Q1D-U against S1D results:

1. Is short-distance part universal? – yes

$$R(\omega\to 0, x) = 4\pi^2 \frac{\partial^2}{\partial x^2} \int_0^\infty k\, dk\, \frac{\tanh \pi k}{\cosh^2 \pi k} e^{-(k^2+1/4)x}$$

2. Is Mott-length-scale part universal? – yes

$$R(\omega, x) \approx \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{x - L_M}{2\sqrt{L_M}}\right)\right]$$

3. Are finite-ω corrections universal? – no

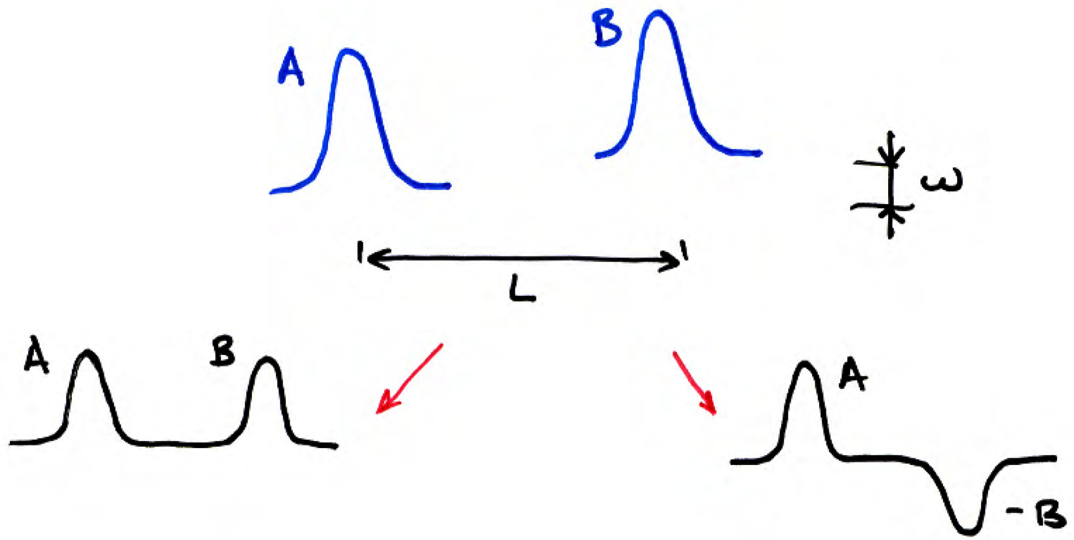
Summary 1 (localization in normal wires)

1. We have obtained a perturbative expansion in ω (including log corrections) of the correlations of local density of states in Q1D wires in the unitary symmetry class

2. We have confirmed the universal properties of S1D / Q1D-U localization
   – for the single-wave-function statistics (known results)
   – at the Mott length scale (new, but expected)
   and studied non-universal corrections in ω (new result)

3. Possible extensions of the method ?
   – dynamical response function $S(\omega, x)$?
   – orthogonal symmetry class?
   These problems are technically more complicated with a sigma model, but: some progress is possible with the Mott hybridization argument !

Improving Mott argument: hybridization with log-normal tails



Assuming log-normal distribution of tails with one-parameter scaling

$$|\psi(r)|^2 = e^{\chi}\,, \qquad dP(\chi) = \frac{1}{2\sqrt{\pi r}} \exp\left[-\frac{(\chi + r)^2}{4r}\right] d\chi$$

$\rightarrow$ reproduces the leading terms of the Q1D-U calculation and suggests new results for $S(\omega, x)$ and for Q1D-O case

Log-normal Mott, exact results, and new conjectures

DOS correlation function $R(\omega, x) = \nu^{-2}\langle \rho_E(0)\rho_{E+\omega}(x)\rangle$

Model	$R(\omega{=}0, x{\gg}1)$	$\delta R(\omega, x{\gg}1)$	at Mott length
1D	✓ $x^{-3/2}e^{-x/4}$	$\omega^2(L_M - 3x)e^{2x}$	✓ $\frac{1}{2}\left(1 + \mathrm{erf}\,\frac{x-L_M}{2\sqrt{x}}\right)$
Q1D-U	✓ $x^{-3/2}e^{-x/4}$	✓ $\omega^2(L_M - 3x)^2 e^{2x}$	✓ $\frac{1}{2}\left(1 + \mathrm{erf}\,\frac{x-L_M}{2\sqrt{x}}\right)$
Q1D-O	$x^{-3/2}e^{-x/4}$	$\omega e^{-x/2}$	$\frac{1}{2}\left(1 + \mathrm{erf}\,\frac{x-L_M}{2\sqrt{x}}\right)$

Dynamical response function $S(\omega, x) = \nu^{-2}\langle G^R_E(0,x)G^A_{E+\omega}(x,0)\rangle$

Model	$S(\omega{=}0, x{\gg}1)$	$\delta S(\omega, x{\gg}1)$	at Mott length
1D	✓ $x^{-3/2}e^{-x/4}$	$-\omega^2(L_M - 3x)e^{2x}$	✓ $-\frac{\exp\left[-\frac{(x-L_M)^2}{4x}\right]}{2\sqrt{\pi x}}$
Q1D-U	$x^{-3/2}e^{-x/4}$	$-\omega^2(L_M - 3x)^2 e^{2x}$	$-\frac{\exp\left[-\frac{(x-L_M)^2}{4x}\right]}{2\sqrt{\pi x}}$
Q1D-O	$x^{-3/2}e^{-x/4}$	$-\omega e^{-x/2}$	$-\frac{\exp\left[-\frac{(x-L_M)^2}{4x}\right]}{2\sqrt{\pi x}}$

✓ – earlier exact results (green)

✓ – our sigma-model calculations (red)

Summary 2 (Mott hybridization with log-normal tails)

1. Hybridization of log-normally distributed tails: an easy approximation to study localized states (much simpler than exact methods)

2. New results (conjectures) for quantum wires in the orthogonal symmetry class and for the dynamical response function

   Possible extensions:

   – away from one-parameter scaling (strong disorder)
   – to higher dimensions
   – to wires with a finite number of channels (crossover from $N = 1$ to $N = \infty$)
   – to contacts between Anderson insulators and superconductors

Localization in SN junctions (three symmetry classes: B,C,D)

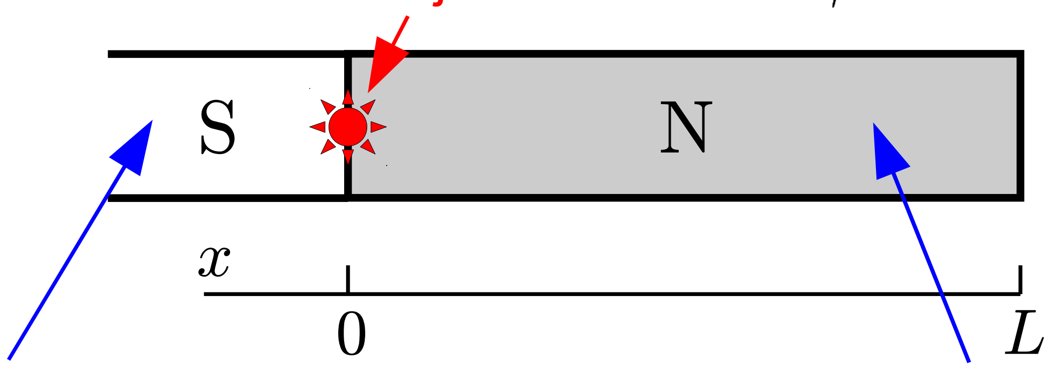
Classes C and D:

conventional superconductor (class CI)

disordered normal wire: time-reversal symmetry broken, spin-rotation symmetry either preserved (→ NS junction of class C) or broken (→ NS junction of class D)

Class B:

Majorana fermion: $\gamma = u\Psi^\dagger + v\Psi = \gamma^\dagger$



quasi-1D topological superconductor: time-reversal and spin-rotation symmetries broken, B symmetry class

disordered normal wire: time-reversal and spin-rotation symmetries broken, A (unitary) symmetry class

Superconducting proximity vs. localization in the case of a broken time-reversal symmetry

- At the quasiclassical level, no proximity effect if the time-reversal symmetry is broken
- Localization helps: proximity survives at the length scale ξ and at the energy scale Δ_ξ (around the Fermi level)
- Due to Andreev reflection, one-point average $\langle \rho_E(x) \rangle$ already exhibits localization
- Calculation is performed along the same lines of the sigma model as in the normal-wire case. Interface with the superconductor becomes a boundary condition for the sigma model

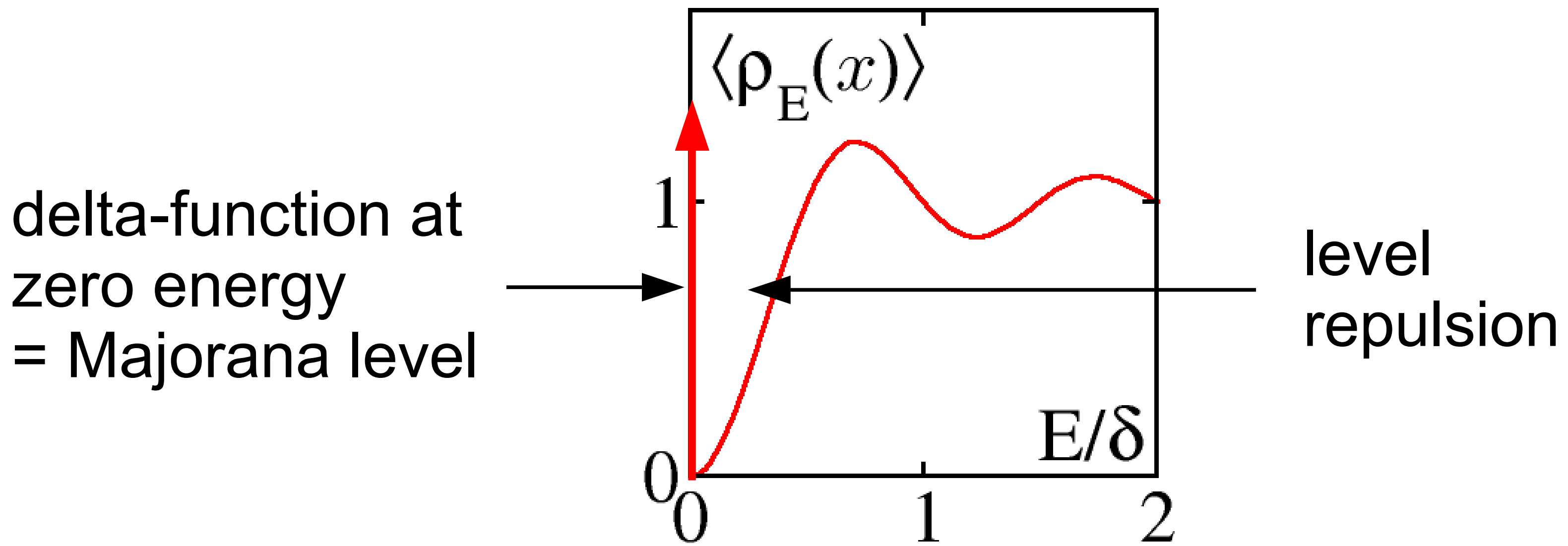
Results in the Majorana case (class B): RMT limit

1. Short-wire limit ($L \ll \xi$):

$$\langle \rho_E(x) \rangle = 1 - \frac{\sin(2\pi E/\delta)}{2\pi E/\delta} + \delta(E/\delta)$$

where δ is the level spacing in the wire

– known result from the random-matrix theory (RMT)



Results in the Majorana case (class B): long-wire limit

2. Long-wire limit ($L \gg \xi$):

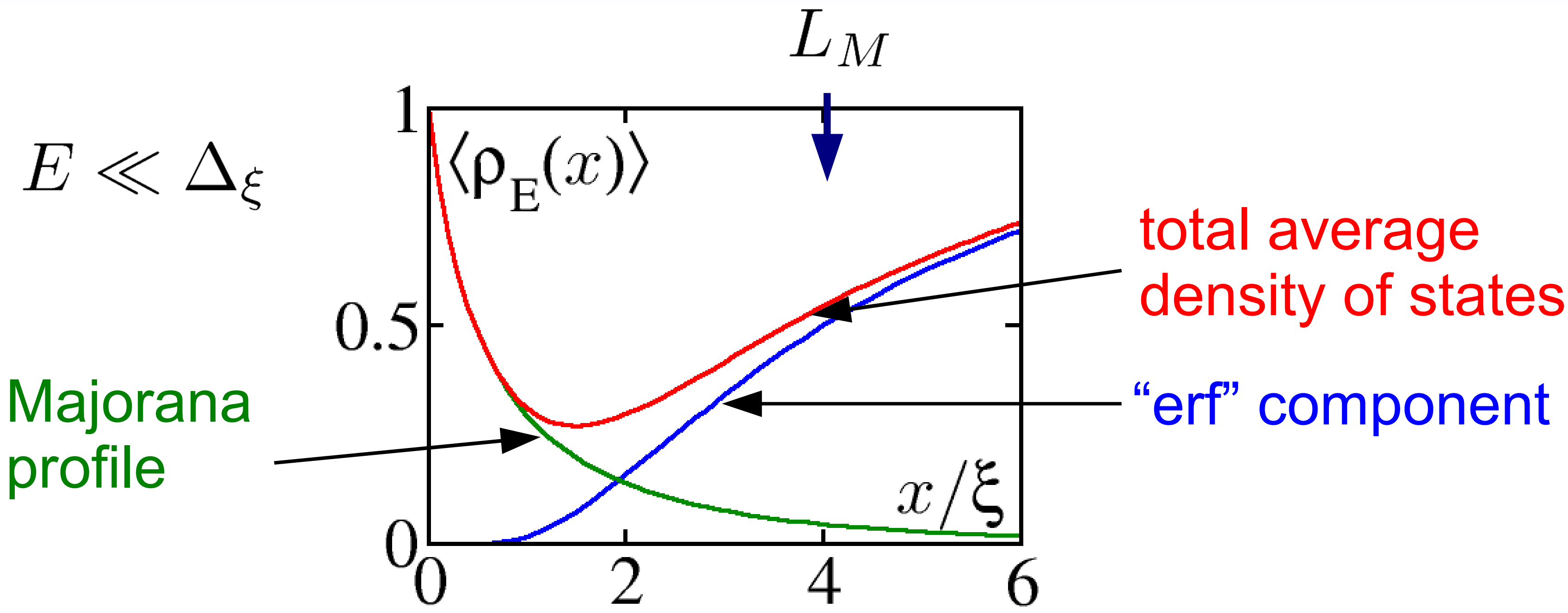
$$\langle \rho_E(x) \rangle = \Phi_M(x) \left[\pi \delta(E/\Delta_\xi) + 1 \right] + \frac{1}{2} \left[1 + \mathrm{erf}\left(\frac{x - L_M}{2\sqrt{x\xi}} \right) \right]$$

where the (average) profile of the Majorana state is

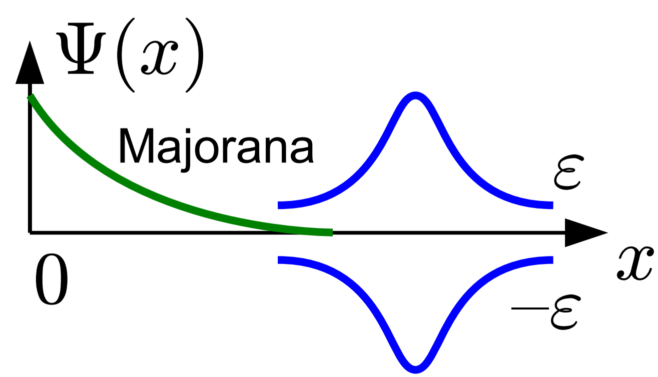
$$\Phi_M(x) = 2\pi \int_0^\infty k\, dk\, \frac{\sinh \pi k}{\cosh^2 \pi k} (k^2 + 1/4)\, e^{-(x/\xi)(k^2+1/4)}$$

and the Mott length is $L_M = 2\xi \ln(\Delta_\xi / E)$

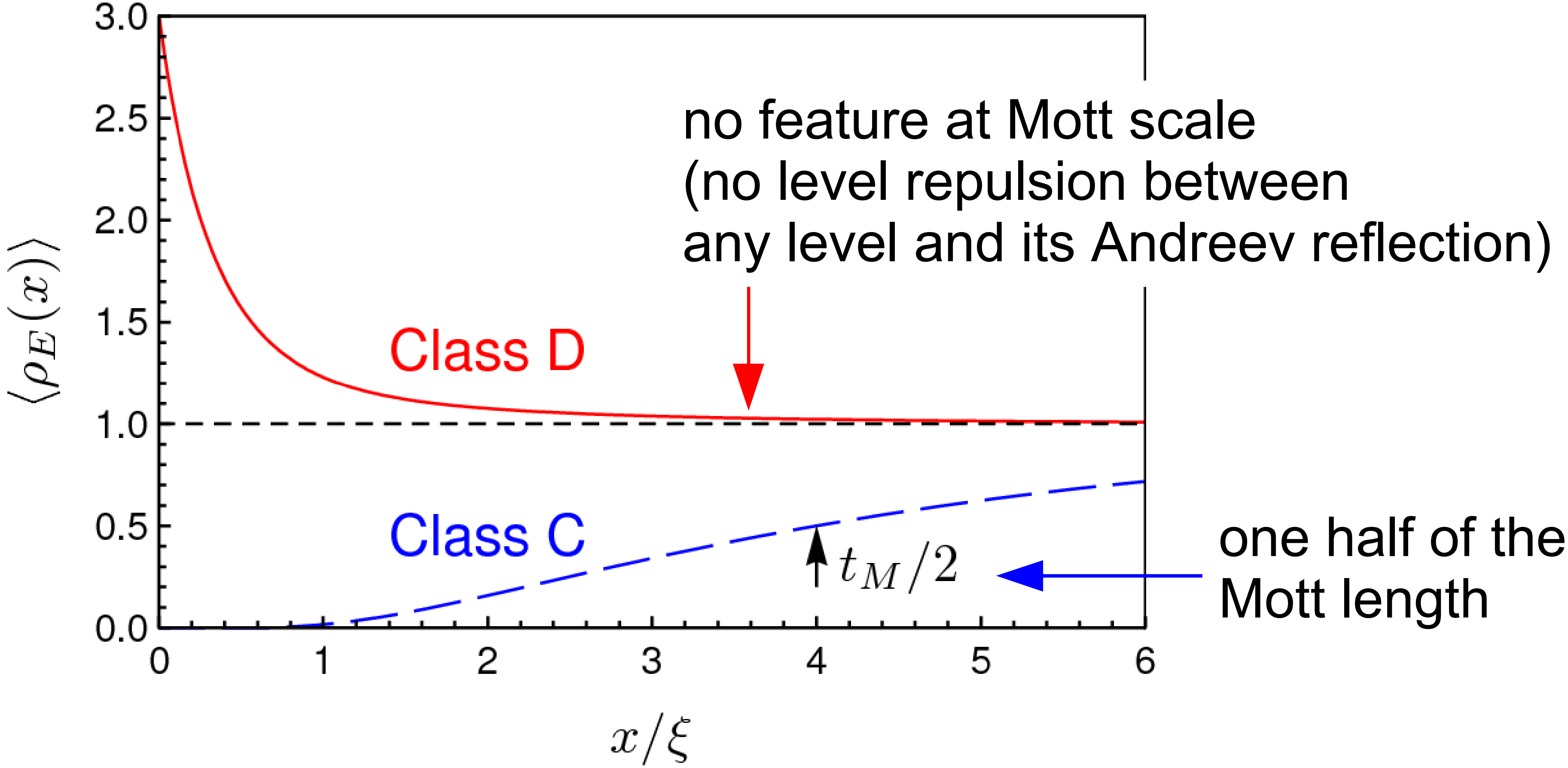
Long-wire limit: interpretation



Can be explained in terms of the "Mott hybridization" of the Majorana state with localized states in the wire



Results for classes C and D (long-wire limit)



these results are obtained by a direct calculation with boundary conditions corresponding to C and D symmetry classes – and then explained in terms of Mott hybridization

Summary 3 (NS junctions)

- An analytic solution for the localization of a Majorana fermion in a disordered NS junction
- Most probably, our results in the $E \ll \Delta_\xi$ limit are also universally valid for wires with orthogonal and symplectic symmetries and for wires with a finite number of channels (assuming weak disorder)
- The same approach is applicable to symmetry classes C and D : it describes the proximity effect in the localized regime (in wires with broken time-reversal symmetry)
- The results of the sigma-model calculations can be interpreted in terms of Mott hybridization of localized states